

# Impact of redundant checks on the LP decoding thresholds of LDPC codes\*

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## Abstract

Feldman *et al.* (2005) asked whether the performance of the Linear Programming (LP) decoder can be improved by adding redundant parity checks to tighten the LP relaxation. We prove in this paper that for LDPC codes, even if we include all redundant parity checks, asymptotically there is no gain in the LP decoder threshold on the Binary Symmetric Channel (BSC) under certain conditions on the base Tanner graph. First, we show that if the Tanner graph has bounded check-degree and satisfies a condition which we call *asymptotic strength*, then including high degree redundant parity checks in the LP does not significantly improve the threshold of the LP decoder in the following sense: for each constant  $\delta > 0$ , there is a constant  $k > 0$  such that the threshold of the LP decoder containing all redundant checks of degree at most  $k$  improves by at most  $\delta$  upon adding to the LP all redundant checks of degree larger than  $k$ . We conclude that if the graph satisfies an additional condition which we call *rigidity*, then including all redundant checks does not improve the threshold of the base LP. We call the graph *asymptotically strong* if the LP decoder corrects a constant fraction of errors even if the log-likelihood-ratios of the correct variables are arbitrarily small. By building on a construction due Feldman *et al.* (2007) and its recent improvement by Viderman (2013), we show that asymptotic strength follows from sufficiently large variable-to-check expansion. We also give a geometric interpretation of asymptotic strength in terms pseudocodewords. We call the graph *rigid* if the minimum weight of a sum of check nodes involving a cycle tends to infinity as the block length tends to infinity. Under the assumptions that the graph girth is logarithmic and the minimum check degree is at least 3, rigidity is equivalent to the *nondegeneracy* property that adding at least logarithmically many checks does not give a constant weight check. We argue that nondegeneracy is a typical property of random check-regular Tanner graphs.

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# 1 Introduction

A Low Density Parity Check (LDPC) code is a linear code whose parity check matrix is sparse. LDPC codes were discovered by Gallager [Gal62] in 1962 who used the sparsity of the parity check matrix to design various iterative decoding algorithms with good performance. The parity check matrix of a LDPC is represented by a bipartite graph, called a Tanner graph [Tan81], between a set of variable nodes and set of check nodes. The past two decades saw a growing number of research results related to LDPC codes and their iterative decoding algorithms (see [RU08] for a comprehensive account). Graph properties such as good girth [Gal62, Tan81] and expansion [SS96] play a central role in designing good LDPC codes with efficient iterative decoding algorithms.

Linear Programming (LP) decoding of linear codes was introduced by Feldman *et al.* [Fel03, FWK05] as a good-performance low-complexity relaxation of Maximum Likelihood (ML) decoding. In the past decade, the good performance of LP decoding of LDPC codes was established in a sequence of papers which lead again to good girth and expansion as desirable properties of the underlying Tanner graph. The LP decoder corrects a constant fraction of errors if the graph has sufficiently large expansion [FMS+07, DDKW08, Vid13]. Moreover, the LP decoder of certain expander codes achieves the capacity of a wide class of binary-input memoryless symmetric channels [FS05]. Lower bounds on the LP decoding thresholds of LDPC codes were obtained in [KV06, ADS12] under the assumption that the graph has a logarithmic girth, and upper bounds were obtained in [VK06]. The LP decoding polytope was independently discovered by Koetter and Vontobel [KV03] in the context of graph covers of Tanner graphs and iterative decoding algorithms. The link between LP decoding and iterative decoding algorithms, in particular the min-sum algorithm, was further investigated in [VK04, ADS12].

Feldman *et al.* [Fel03, FWK05] asked whether the performance of the LP decoder can be improved by tightening the LP relaxation. Namely, they proposed two natural approaches to tighten the LP: (1) adding *redundant parity checks* and (2) *Lifting techniques*. Another tightening technique based on merging nodes was explored by Burshtein and Goldenberg [BG11].

This paper is about the first approach. Including redundant parity checks does not affect the code but adds new constraints to the LP. The problem of appropriately selecting redundant checks to be added to the LP without sacrificing its efficiency was investigated in [TS08, MWT09]. Even though simulation results suggest that redundant checks improve the LP decoder performance [FWK05, TS08, MWT09], we argue in this paper that asymptotically there is no gain in terms of the LP decoder threshold on the BSC even if we add all redundant checks, assuming that the base Tanner graph has bounded check-degree and satisfies two natural conditions which we call asymptotic strength and rigidity. The required conditions are satisfied if in addition to sufficiently good expansion and girth, the graph has a nondegeneracy property, which holds with high probability for random check-regular graphs.

As for the lifting techniques, a recent result of Ghazi and Lee [GL14] shows that extensions of the LP decoder based on Sherali-Adams and Lasserre hierarchies do not significantly improve the error correction capabilities of the LP decoder if the graph is a good expander.

The common theme between our result and the result of [GL14] is that if the base LP has “certain desirable or typical properties” then it is “hard to make it asymptotically better”. Related to this theme is the other extreme of *geometrically perfect codes*, which are by definition codes for which the LP resulting from adding all redundant checks is equivalent to ML decoding (see Section 1.2); such codes are asymptotically bad by a recent result due to Kashyap [Kas08].

On the positive side, our negative results suggest studying the LP decoding limits in the frame-

work of the dual code containing all redundant check nodes. This framework is appealing since it is independent of a particular Tanner graph representation of the code.

The proof of our main result is based on a careful analysis of the dual LP. We use the dual witness and hyperflow structures [FMS+07, DDKW08] and the fact that the existence of such structures is necessary for LP decoding success [BGU14]. We also use the notion of acyclic hyperflows and the LP excess technique [BGU14]. To establish the relation between asymptotic strength and expansion, we build on the dual witness construction in [FMS+07, Vid13]. Our probabilistic analysis of nondegeneracy is based on the work of Calkin [Cal97].

In the remainder of this introductory section, we give background material on Tanner graphs, redundant checks and LP decoding. Then, we formally state our results in Section 1.3 and we give a detailed outline of the rest of the paper in Section 1.4.

## 1.1 Tanner graphs and redundant checks

A *Tanner graph*  $G = (V, C, E)$  is an undirected bipartite graph between a set  $V$  of *variable nodes* and a set  $C$  of *check nodes*, where  $E$  is the set of edges. If  $i \in V$  is a variable node, we will denote by  $N(i)$  the check neighborhood of  $i$ , i.e., the set of check nodes adjacent to  $i$ . Similarly, if  $j \in C$  is check node,  $N(j)$  is the set of variable nodes adjacent to  $j$ . Unless otherwise specified, we assume throughout the paper that  $V = \{1, \dots, n\}$ , where  $n \geq 1$  is the *block length*. We assume also that the degree of each check node is at least one. The *linear code*  $Q = Q_G$  associated with  $G$  is the  $\mathbb{F}_2$ -linear code  $Q \subset \mathbb{F}_2^n$  whose parity check matrix is the biadjacency matrix of  $G$ . That is,  $Q$  is the set of all binary strings  $x \in \mathbb{F}_2^n$  such that  $\sum_{i \in N(j)} x_i = 0$  for each  $j \in C$ .

Given a tanner graph  $G = (V, C, E)$ , the *Tanner graph of all redundant checks*  $\overline{G}$  associated with  $G$  is defined as follows. A *redundant check* of  $G$  is a nonzero  $\mathbb{F}_2$ -linear combinations of checks of  $G$ , thus the redundant checks are in one-to-one correspondence with the nonzero elements of the dual code  $Q^\perp$ . The graph  $\overline{G}$  is obtained from  $G$  by adding all redundant checks to  $G$ . That is,  $\overline{G} = (V, \overline{C}, \overline{E})$ , where  $\overline{C} = Q^\perp - \{0\}$  and  $i \in V$  is connected to  $c \in Q^\perp$  iff  $c_i = 1$ .

We are also interested in the following graded subgraphs of  $\overline{G}$ . Given  $G = (V, C, E)$  and an integer  $k$ , let  $\overline{G}^k$  be the *Tanner graph of redundant checks of degree at most  $k$* . That is,  $\overline{G}^k = (V, \overline{C}^k, \overline{E}^k)$  is the subgraph of  $\overline{G}$  induced on  $V$  and the set  $\overline{C}^k$  of nonzero checks of degree at most  $k$ , i.e.,  $\overline{C}^k = \{c \neq 0 \in Q^\perp : \text{weight}(c) \leq k\}$ . Thus, if  $d$  is the maximum degree of a check node in  $G$ , we have the nested sequence of Tanner graphs  $G \subset \overline{G}^d \subset \overline{G}^{d+1} \subset \dots \subset \overline{G}^n = \overline{G}$ , all defining the same code  $Q$ . Throughout this paper, we are interested in base Tanner graphs where the *maximum check degree  $d$*  is bounded.

## 1.2 Linear programming decoder

Let  $G = (V, C, E)$  be a Tanner graph and  $Q \subset \mathbb{F}_2^n$  the associated code. Consider transmitting a codeword of  $Q$  over the  $\epsilon$ -BSC (*Binary Symmetric Channel*), which on input  $x \in \mathbb{F}_2^n$  outputs  $y \in \mathbb{F}_2^n$  by flipping each bit of  $x$  independently with probability  $\epsilon$ . The *ML (Maximum Likelihood)* decoder is given by  $\hat{x}_{\text{ML}} = \operatorname{argmax}_{x \in Q} p_{Y|X}(y|x)$ . Let  $\gamma \in \mathbb{R}^n$  be the *LLR (Log-Likelihood-Ratio)* vector of  $y$ :  $\gamma_i = \log \left( \frac{p_{Y_i|X_i}(y_i|0)}{p_{Y_i|X_i}(y_i|1)} \right) = (-1)^{y_i} \log \frac{1-\epsilon}{\epsilon}$  for  $i = 1, \dots, n$ . In terms of  $\gamma$ , the ML decoder is given by

$$\hat{x}_{\text{ML}} = \operatorname{argmin}_{x \in Q} \langle x, \gamma \rangle, \quad (1)$$

where  $\langle x, \gamma \rangle := \sum_i x_i \gamma_i$ . For general linear codes, the ML decoding problem is NP-hard [BMVT78]. Feldman *et al.* [Fel03, FWK05] introduced the approach of *LP (Linear Programming)* decoding, which is based on relaxing the optimization problem on  $Q$  into an LP. Due to the linearity of the objective function  $\langle x, \gamma \rangle$ , optimizing over  $Q$  is equivalent to optimizing over the convex polytope  $\text{conv}(Q) \subset \mathbb{R}^n$  spanned by the convex combinations of the codewords in  $Q$ :

$$\hat{x}_{\text{ML}} = \underset{x \in \text{conv}(Q)}{\text{argmin}} \langle x, \gamma \rangle. \quad (2)$$

The idea of Feldman is to relax  $\text{conv}(Q)$  into a larger lower-complexity polytope. For each check node  $j \in C$ , define the local code  $Q_j$  consisting of all vectors  $x \in \{0, 1\}^n$  satisfying check  $j$ , thus  $Q = \bigcap_{j \in C} Q_j$ . Let

$$P(G) := \bigcap_{j \in C} \text{conv}(Q_j) \supset \text{conv}\left(\bigcap_{j \in C} Q_j\right) = \text{conv}(Q). \quad (3)$$

The polytope  $P(G)$  depends on the Tanner graph representation of the code and it is called the *fundamental polytope* of  $G$ . The *LP decoder* is the relaxation of the ML decoder given by

$$\hat{x}_{\text{LP}} = \underset{x \in P(G)}{\text{argmin}} \langle x, \gamma \rangle. \quad (4)$$

The relaxed LP can be efficiently solved due to the low complexity of  $P(G)$ . More generally, (1) and (4) define the ML and LP decoder for an arbitrary LLR vector  $\gamma \in \mathbb{R}^n$ . If  $\gamma$  is as above associated with a binary vector  $y$ , we ignore without loss of generality the constant  $\log \frac{1-\epsilon}{\epsilon}$  and we normalize  $\gamma$  so that  $\gamma = (-1)^y$ .

It is appropriate to mention at this stage geometrically perfect codes. A linear code  $Q \subset \mathbb{F}_2^n$  is called *geometrically perfect* [BG86, Kas08] if the LP relaxation corresponding to the full dual code is exact, i.e.,  $P(\bar{G}) = \text{conv}(Q)$ , where  $G$  is any Tanner graph of  $Q$ . Examples of such codes are tree codes and cycle codes. Geometrically perfect codes are classified in [BG86] based on Seymours matroid decomposition theory [Sey80], but they are unfortunately asymptotically bad in the sense that their minimum distance does not grow linearly with the block length [Kas08].

We are interested in LP thresholds over the BSC as the block length  $n$  tends to infinity. That is, we have an *infinite family of Tanner graphs*  $\mathcal{G} = \{G_n\}_n$ , where  $G_n = (V_n, C_n, E_n)$  is a Tanner graph on  $n$  variable nodes, i.e.,  $V_n = \{1, \dots, n\}$ . Define the *LP-threshold*  $\xi_{\text{LP}}(\mathcal{G})$  of  $\mathcal{G}$  to be the supremum of  $\epsilon \geq 0$  such that the error probability of the LP decoder of  $G_n$  over the  $\epsilon$ -BSC goes to zero as  $n$  tends to infinity, i.e.,

$$\xi_{\text{LP}}(\mathcal{G}) = \sup\{\epsilon \geq 0 : \Pr_{\epsilon\text{-BSC}}[\text{LP decoder of } G_n \text{ fails}] = o(1)\}.$$

As in previous work [FWK05], we assume without loss of generality that the all-zeros codeword was transmitted and that the LP decoder fails if zero is not the unique optimal solution of the LP.

Finally, given an infinite family of Tanner graphs  $\mathcal{G} = \{G_n\}_n$ , we are interested in the resulting family  $\bar{\mathcal{G}} := \{\bar{G}_n\}_n$  of Tanner graphs obtained by adding all redundant checks. Moreover, if  $k : \mathbb{N}^+ \rightarrow \mathbb{R}$ , we are also interested in the family  $\bar{\mathcal{G}}^k := \{\bar{G}_n^{k(n)}\}_n$  of Tanner graphs obtained by adding all redundant checks of degree at most  $k$ .

### 1.3 Summary of results

Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs of bounded check degree. We show that if  $\mathcal{G}$  satisfies a condition which we call *asymptotic strength*, then including high degree redundant checks in the LP does not improve the threshold in the sense that for each constant  $\delta > 0$ , there is a constant  $k > 0$  such that  $\xi_{LP}(\overline{\mathcal{G}}^k) \geq \xi_{LP}(\overline{\mathcal{G}}) - \delta$ . We conclude that if  $\mathcal{G}$  satisfies an additional condition which we call *rigidity*, then including all redundant checks does not improve the threshold of the base LP in the sense that  $\xi_{LP}(\mathcal{G}) = \xi_{LP}(\overline{\mathcal{G}})$ . We call the graph *asymptotically strong* if the LP decoder corrects a constant fraction of errors even if the LLR values of the correct variables are arbitrarily small. We show that the asymptotic strength condition follows from expansion. We call the graph *rigid* if the minimum weight of a sum of check nodes involving a cycle tends to infinity as the block length tends to infinity. We note that under the assumptions that the girth of  $G_n$  is  $\Theta(\log n)$  and the minimum check degree is at least 3, rigidity is equivalent to the property that adding  $\Omega(\log n)$  checks does not give  $O(1)$  weight checks, which we argue is a typical property of random check-regular Tanner graphs.

**Definition 1.1 (Asymptotically strong Tanner graphs)** *Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs. We call  $\mathcal{G}$  asymptotically strong if for each (small) constant  $\beta > 0$ , there exists a constant  $\alpha > 0$  such that for each  $n$  and each error vector  $y \in \{0, 1\}^n$  of weight at most  $\alpha n$ , the LP decoder of  $G_n$  succeeds on the asymmetric LLR vector  $\gamma(y, \beta) \in \mathbb{R}^n$  given by*

$$\gamma_i(y, \beta) = \begin{cases} -1 & \text{if } y_i = 1 \\ \beta & \text{if } y_i = 0, \end{cases}$$

for  $i = 1, \dots, n$ .

Although asymmetry in the LLR vector might seem unnatural at this point, we start with this definition of asymptotic strength because it gives flexibility in the analysis. We give later an equivalent definition in terms of pseudocodewords (Theorem 8.1).

**Theorem 1.2 (High degree redundant checks do not improve LP threshold)** *Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs such that each check node has degree at most  $d$ , where  $d$  is a constant. Assume that  $\mathcal{G}$  is asymptotically strong. Then:*

- a) *For any small constant  $\delta > 0$ , there exists a sufficiently large constant  $k \geq d$  (dependent on  $\delta$  and independent of  $n$ ) such that  $\xi_{LP}(\overline{\mathcal{G}}^k) \geq \xi_{LP}(\overline{\mathcal{G}}) - \delta$ .*
- b) *If  $k(n)$  is a real valued function of  $n$  such that  $k(n) = \omega(1)$  (i.e.,  $k(n)$  tends to infinity as  $n$  tends to infinity), then  $\xi_{LP}(\overline{\mathcal{G}}^{k(n)}) = \xi_{LP}(\overline{\mathcal{G}})$ .*

The proof of Theorem 1.2 uses the LP excess lemma [BGU14] and the notion of primitive hyperflows which we define at the end of this section.

Feldman *et al.* [FMS+07] argued that expansion implies that the LP decoder corrects a positive fraction of errors. The link between the expansion of a Tanner graph and the error correction capabilities of the underlying code was discovered by Sipser and Spielman [SS96] in the context of iterative decoding algorithms. Recently, Viderman [Vid13] simplified the argument of [FMS+07] and improved its dependency on the expansion parameter. By building on the construction in [FMS+07, Vid13], we show that graphs with good expansion are asymptotically strong.

A Tanner graph  $G = (V, C, E)$  is called an  $(\varepsilon n, \kappa)$ -*expander* if for each subset  $S \subset V$  of variable nodes of size at most  $\varepsilon n$ , we have  $|N(S)| \geq \kappa|S|$ , where  $N(S)$  is the set of (check) nodes adjacent to  $S$ .

**Theorem 1.3 (Expansion implies asymptotic strength)** *Let  $d_v > 0, \varepsilon > 0$  and  $\delta > \frac{2}{3}$  be constants such that  $d_v$  is an integer and  $\delta d_v$  is an integer. Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs with regular variable degree  $d_v$  and bounded check degree. If  $G_n$  is an  $(\varepsilon n, \delta d_v)$ -expander for each  $n$ , then  $\mathcal{G}$  is asymptotically strong.*

It is known that redundant check nodes obtained by acyclic sums of check nodes do not tighten the polytope [Fel03, VK05, BG11], which motivates the following definitions.

**Definition 1.4 (Cyclic sums of checks)** *Let  $G = (V, C, E)$  be a Tanner graph. We call a subset of check nodes  $S \subset C$  **cyclic** if the graph induced by  $G$  on  $S$  contains a cycle.*

*Define  $\Delta(G)$  to be the **minimum weight of the sum of a cyclic subset of check nodes** of  $G$ . More formally, let  $Q \subset \mathbb{F}_2^n$  be the code associated with  $G$ . For each check  $j \in C$ , let  $z_j \in Q^\perp$  be the vector in the dual code associated with  $j$ . Then*

$$\Delta(G) := \min\{\text{weight}(\sum_{j \in S} z_j) : S \subset C \text{ cyclic}\}.$$

**Definition 1.5 (Rigid Tanner graphs)** *We call an infinite family  $\mathcal{G} = \{G_n\}_n$  of Tanner graphs **rigid** if the minimum weight of a sum of check nodes involving a cycle tends to infinity as the block length tends to infinity. More formally,  $\mathcal{G}$  is rigid if  $\Delta(G_n) = \omega(1)$ .*

**Remark 1.6** If  $\mathcal{G}$  is rigid, then the check nodes of  $G_n$  are linearly independent for sufficiently large  $n$  (since any subset of check nodes whose sum is zero must be cyclic).

Accordingly, we obtain the following corollary to Theorem 1.2.

**Corollary 1.7 (Redundant checks do not improve LP threshold)** *Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs of bounded check degree. If  $\mathcal{G}$  is asymptotically strong and rigid, then  $\xi_{LP}(\overline{\mathcal{G}}) = \xi_{LP}(\mathcal{G})$ .*

It is not hard to see that  $\omega(1)$ -girth is a necessary condition for rigidity. Unfortunately, random graphs have  $O(1)$ -girth, thus they are not necessarily rigid. In general,  $\Theta(\log n)$ -girth is a desirable property of a Tanner graph in the context of LP decoding [FWK05] and iterative decoding [Gal62, Tan81]. Random graphs with good girth are typically constructed by breaking the cycles of a random graph. We note that for graphs with  $\Theta(\log n)$ -girth and minimum check degree at least 3, rigidity is equivalent to a simpler nondegeneracy condition which we define below.

**Definition 1.8 (Nondegeneracy)** *Call an  $m \times n$  matrix  $M \in \mathbb{F}_2^{m \times n}$   $(s, k)$ -nondegenerate if the sum of any subset of at least  $s$  rows of  $M$  has weight larger than  $k$ . We call a Tanner graph  $G$   $(s, k)$ -nondegenerate if its  $m \times n$  biadjacency matrix is  $(s, k)$ -nondegenerate, where  $m$  is the number of check nodes and  $n$  is the number of variable nodes.*

For instance, full row rank corresponds to  $(1, 0)$ -nondegeneracy.

**Lemma 1.9 (Rigidity versus girth and nondegeneracy)** *Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs of bounded check degree. If  $\mathcal{G}$  is rigid, then  $\text{girth}(G_n) = \omega(1)$ . On the other hand, if  $\text{girth}(G_n) = \Theta(\log n)$  and the minimum check degree of  $\mathcal{G}$  is at least 3 (i.e., for all  $n$ , each check node of  $G_n$  has degree at least 3), then the following are equivalent:*

- i) (Rigidity)  $\mathcal{G}$  is rigid
- ii) (Nondegeneracy) *For each constant  $c > 0$ ,  $G_n$  is  $(c \log n, \omega(1))$ -nondegenerate. That is, for each constant  $c > 0$ , the minimum weight of a sum of at least  $c \log n$  checks nodes tends to infinity as  $n$  increases.*

We argue that nondegeneracy is a typical property of random check-regular Tanner graphs. Namely, we show that random check-regular graphs are  $(c \log n, \omega(1))$ -nondegenerate with high probability if  $m \leq \beta_d n$ , where  $d$  the check degree and  $\beta_d$  is Calkin's threshold as given in Definition 7.1 ( $\beta_d$  is a threshold close to 1, e.g.,  $\beta_3 \sim 0.8895$ ,  $\beta_4 \sim 0.967$  and  $\beta_5 \sim 0.989$ ).

**Lemma 1.10 (Random check-regular graphs are nondegenerate)** *Let  $d, m$  and  $n$  be integers such that  $d \geq 3$  and  $1 \leq m < \beta_d n$ . Consider a random  $m \times n$  matrix  $M \in \mathbb{F}_2^{m \times n}$  constructed by independently choosing each of the  $m$  rows of  $M$  uniformly from the set of vectors in  $\mathbb{F}_2^n$  of weight  $d$ . Then for any constant  $c > 0$  and any function  $k(n)$  of  $n$  such that  $k(n) = o(\log \log n)$ ,  $M$  is  $(c \log n, k(n))$ -nondegenerate with high probability.*

We establish the claim by adapting an argument used by Calkin [Cal97] to show that if  $m < \beta_d n$ , then  $M$  has full row rank with high probability. The ensemble of random check-regular graphs is attractive from a probabilistic analysis standpoint, but it typically gives irregular graphs with constant girth. We believe that good girth and variable-regularity do not increase the odds of degeneracy; we conjecture that the statement of Lemma 1.10 extends to the ensemble of regular  $\Theta(\log n)$ -girth Tanner graphs (see Section 10).

We also prove the following general results about LP decoding which might be of independent interest:

- **(Primitive hyperflows)** We give a simple necessary and sufficient condition for the success of the LP decoding when all redundant checks are included in the LP. The condition is in terms of the existence of a hyperflow (see Definition 2.1) which is primitive in the sense that all the variables in error have zero outflow and all the correct variables have zero inflow (Theorem 4.2). This characterization is essential to the proof of Theorem 1.2.
- **(Pseudocodewords interpretation of asymptotic strength)** We note that the notion of asymptotic strength has the following geometric interpretation in terms of pseudocodewords:  $\mathcal{G} = \{G_n\}_n$  is asymptotically strong iff for each nonzero pseudocodeword  $x \in P(G_n)$ , to attain a positive fraction of  $\sum_i x_i$ , we need a least linear number of coordinates of  $x$ . That is, for each  $\theta > 0$ , there exists  $\alpha > 0$  such that for each  $n$  and each nonzero pseudocodeword  $x \in P(G_n)$ , the sum of the largest  $\lfloor \alpha n \rfloor$  coordinates of  $x$  is less than  $\theta \sum_i x_i$  (Theorem 8.1).
- **(Asymptotic strength and LP decoding with help)** Assume that we are allowed to flip at most a certain number of bits of the corrupted codeword to help the LP decoder on the BSC. We argue that if the Tanner graph is asymptotically strong, allowing a sublinear number of help bits does not improve the LP threshold (Theorem 9.2). This result, although a negative statement, has potential constructive applications as it weakens the dual witness requirement for LP decoding success.

- **(LP deficiency lemma)** We give a converse of the LP excess lemma [BGU14]. Namely, we show how to trade LP-deficiency for crossover probability (Lemma 9.3) and we use the LP deficiency lemma to establish the above result on LP decoding with help.

## 1.4 Outline

In Section 2, we give background material on graph structures whose existence is necessary and sufficient for LP decoding success: dual witness, hyperflows and acyclic hyperflows. To warm up, we highlight in Section 3 a simple classical argument, which shows that high density codes have zero thresholds on the BSC. The key starting point of our proof is the above-mentioned special type of hyperflows called primitive hyperflows. We define primitive hyperflows in Section 4 and we argue that their existence is sufficient for LP decoding success when all redundant checks are included in the LP. In Section 5, we show that for asymptotically strong codes with bounded-check degree, high degree checks do not improve the threshold (Theorem 1.2). Then we conclude that adding all redundant checks does not improve the threshold if the graph is additionally rigid (Corollary 1.7). In Section 6, we study the relation between expansion and asymptotic strength (Theorem 1.3). In Section 7, we study the rigidity and the related nondegeneracy properties (Lemmas 1.9 and 1.10). In Section 8, we give the above-mentioned pseudocodewords interpretation of asymptotic strength. In Section 9, we give an application of asymptotically strong codes in the context of the above-mentioned problem of LP decoding with help bits. Finally, we conclude in Section 10 with a discussion of the asymptotic strength condition, the rigidity condition and the limits of LP decoding on the BSC.

## 2 LP decoding success, dual witness and hyperflow

In this section we summarize various dual characterizations of LP decoding success that will be used in this paper. The notion of dual witness was introduced in [FMS+07] as a sufficient condition for LP decoding success. The necessity of the existence of a dual witness for LP decoding success was established in [BGU14]. A special type of dual witnesses called hyperflows was introduced in [FMS+07, DDKW08]. The equivalence between the existence of a hyperflow and the existence of a dual witness was established in [DDKW08]. The notion of a hyperflow was further simplified in [BGU14] who argued that the the existence of an acyclic hyperflow is equivalent to the existence of a hyperflow.

**Definition 2.1 ([FMS+07, DDKW08]) (Dual witness, Hyperflow, and WDG)** *Consider a Tanner graph  $G = (V, C, E)$  and an LLR vector  $\gamma \in \mathbb{R}^V$ . A **dual witness** for  $\gamma$  in  $G$  is a function  $w : E \rightarrow \mathbb{R}$  satisfying the inequalities in (a) and (b) below.*

- a) **Variable nodes inequalities:**  $F_i(w) < \gamma_i$ , for each variable  $i \in V$ , where  $F(w) \in \mathbb{R}^V$  is given by

$$F_i(w) := \sum_{j \in N(i)} w(i, j).$$

*We call  $F_i(w)$  the flow at variable node  $i$  associated with  $w$ .*

- b) **Check nodes inequalities:** for each check  $j \in C$  and all distinct variables  $i \neq i' \in N(j)$ ,  $w(i, j) + w(i', j) \geq 0$ .



A dual witness  $w : E \rightarrow \mathbb{R}$  is called a **hyperflow** if, instead of (b), it satisfies the following stronger check nodes inequalities.

- c) **Hyperflow check nodes inequalities:** for each check  $j \in C$ , there exists  $P_j \geq 0$  and a variable  $i \in N(j)$  such that  $w(i, j) = -P_j$  and  $w(i', j) = P_j$ , for all  $i' \neq i \in N(j)$ .

A dual witness or a hyperflow  $w$  can be viewed as a **weighted directed graph (WDG)**  $D$  on the vertices  $V \cup C$ , where an arrow is directed from  $i$  to  $j$  if  $w(i, j) > 0$ , an arrow is directed from  $j$  to  $i$  if  $w(i, j) < 0$  and  $i$  and  $j$  are not connected by an arrow if  $w(i, j) = 0$ . The weight of each directed edge connecting  $i \in V$  and  $j \in C$  is  $|w(i, j)|$ . Thus, in terms of  $D$ , the variable nodes inequalities in (a) can be rephrased as follows.

- d) **WDG variable nodes inequalities:**  $F_i^{\text{out}}(w) < F_i^{\text{in}}(w) + \gamma_i$ , for each variable  $i \in V$ , where  $F^{\text{out}}(w), F^{\text{in}}(w) \in \mathbb{R}^V$  are defined as follows.

- $F_i^{\text{out}}(w) := \sum_{j \in \text{Out}_D(i)} |w(i, j)|$  where  $\text{Out}_D(i)$  is the set of check nodes incident to edges outgoing from  $i$ .
- $F_i^{\text{in}}(w) := \sum_{j \in \text{In}_D(i)} |w(j, i)|$  where  $\text{In}_D(i)$  is the set of check nodes incident to edges ingoing to  $i$ .

We call  $F_i^{\text{out}}(w)$  the outflow from variable node  $i$  associated with  $w$  and  $F_i^{\text{in}}(w)$  the inflow to variable node  $i$  associated with  $w$ .

We summarize in the following theorem various equivalent characterizations of LP decoding success.

**Theorem 2.2 ([FMS+07, DDKW08, BGU14]) (Equivalent characterizations of LP decoding success)** Let  $G = (V, C, E)$  be a Tanner graph and  $\gamma \in \mathbb{R}^V$  an LLR vector. Then the following are equivalent:

- i) The LP decoder of  $G$  succeeds on  $\gamma$  (i.e., it returns zero as the unique solution under the assumption that the all-zeros codeword was transmitted).
- ii) There is a dual witness for  $\gamma$  in  $G$ .
- iii) There is a hyperflow for  $\gamma$  in  $G$ .
- iv) There is a hyperflow for  $\gamma$  in  $G$  whose WDG is acyclic.

**Remark 2.3** The fact that (ii) implies (i) follows from [FMS+07], the fact that (i) implies (ii) follows from Theorem 3.2 and Remark 3.3 in [BGU14], the equivalence between (ii) and (iii) follows from Proposition 1 in [DDKW08] and the equivalence between (ii) and (iv) follows from Theorem 3.7 in [BGU14]. Note that the statement of Theorem 3.7 in [BGU14] assumes that  $\gamma$  is an LLR vector of a binary error pattern (i.e.,  $\gamma \in \{-1, 1\}^V$ ), but its proof holds for an arbitrary LLR vector  $\gamma \in \mathbb{R}^V$ .

### 3 High density codes

In this section, we highlight a simple classical argument which shows that high density codes have zero thresholds on the BSC. A statement similar to Lemma 3.1 below appears in Corollary 7 of [VK06] in the context of regular Tanner graphs (with a different but also simple proof). Although not used in the proofs of the results in this paper, we include this lemma since from a broad perspective it is related to the statement of Theorem 1.2, which says that high degree redundant checks are not helpful if the code is asymptotically strong. Unfortunately, the simple proof of Lemma 3.1 does not extend to the setup of high degree redundant checks.

**Lemma 3.1 (High density codes)** *Let  $G = (V, C, E)$  be a Tanner graph such that the minimum degree of a check node is  $d_{\min}$ . Then the LP decoder of  $G$  fails if the number of errors introduced by the BSC is at least  $n/d_{\min}$ . Thus, if  $\mathcal{G}$  is an infinite family of Tanner graphs such that the minimum degree of a check node in  $G_n$  is  $\omega(1)$ , then the threshold  $\xi_{LP}(\mathcal{G}) = 0$ .*

**Proof:** Assume that the all-zeros codeword was transmitted and let  $y \in \{0, 1\}^n$  be the received vector. If the LP decoder of  $G$  correctly decodes  $y$ , then by Theorem 2.2,  $(-1)^y$  has a hyperflow  $w : E \rightarrow \mathbb{R}$ . Consider the WDG  $D$  corresponding to  $w$  and let  $U = \{i : y_i = 1\}$  be the set of variables in error. If  $S \subset V$ , let  $F^{\text{in}}(w; S) := \sum_{i \in S} F_i^{\text{in}}(w)$  be total inflow to  $S$  and  $F^{\text{out}}(w; S) := \sum_{i \in S} F_i^{\text{out}}(w)$  be total outflow from  $S$ . Summing the variable nodes inequalities  $F_i^{\text{out}}(w) < F_i^{\text{in}}(w) + \gamma_i$  over all  $i \in V$ , we get  $F^{\text{out}}(w; V) < F^{\text{in}}(w; V) + |U^c| - |U|$ , i.e.,

$$F^{\text{out}}(w; V) < F^{\text{in}}(w; V) + n - 2|U|. \quad (5)$$

Summing the variable nodes inequalities over all  $i \in U$ , we get  $F^{\text{out}}(w; U) < F^{\text{in}}(w; U) - |U|$ . Since  $F^{\text{out}}(w; U) \geq 0$  and  $F^{\text{in}}(w; U) \leq F^{\text{in}}(w; V)$ , we obtain

$$|U| < F^{\text{in}}(w; V). \quad (6)$$

Finally, the hyperflow check nodes inequalities ((c) in Definition 2.1) imply that

$$(d_{\min} - 1)F^{\text{in}}(w; V) \leq F^{\text{out}}(w; V). \quad (7)$$

Solving for  $|U|$  in (5), (6) and (7), we obtain  $|U| < n/d_{\min}$ . ■

### 4 Redundant checks and primitive hyperflows

We give in this section a simple necessary and sufficient condition for the success of LP decoding when all redundant checks are included in the LP. The condition is in terms of the existence of a primitive hyperflow which we define as a hyperflow such that all the variables in error have zero outflow and all the correct variables have zero inflow. Primitive hyperflows are central to the proof of Theorem 1.2.

**Definition 4.1 (Primitive hyperflow)** *Let  $H = (V, C, E)$  be a Tanner graph,  $\gamma \in \mathbb{R}^V$  an LLR vector and  $w : E \rightarrow \mathbb{R}$  a hyperflow for  $\gamma$  in  $H$ . Consider the WDG  $D$  of  $w$ . We call  $w$  a **primitive hyperflow** if for each variable nodes  $i \in V$ , we have:*

- a) *If  $\gamma_i \leq 0$ , then  $i$  has no outgoing edges in  $D$ , i.e.,  $F_i^{\text{out}}(w) = 0$ .*

b) If  $\gamma_i > 0$ , then  $i$  has no ingoing edges in  $D$ , i.e.,  $F_i^{in}(w) = 0$ .

Note that the WDG of a primitive hyperflow is necessarily acyclic.

**Lemma 4.2 (Redundant checks and primitive hyperflows)** *Let  $G = (V, C, E)$  be a Tanner graph and consider the associated Tanner graph  $\overline{G} = (V, \overline{C}, \overline{E})$  of all redundant check nodes. Let  $\gamma \in \mathbb{R}^V$  be an LLR vector. If the LP decoder of  $\overline{G}$  succeeds on  $\gamma$ , then there is a primitive hyperflow for  $\gamma$  in  $\overline{G}$*

**Proof:** Assume that the LP decoder of  $\overline{G}$  succeeds on  $\gamma$ . By Theorem 2.2, there exists a hyperflow  $w : \overline{E} \rightarrow \mathbb{R}$  for  $\gamma$  in  $\overline{G}$  whose WDG  $D$  is *acyclic*. We will make  $D$  primitive by exploiting the key property of  $\overline{G}$  that its check nodes are in one-to-one correspondence with the nonzero vectors in the dual  $Q^\perp$  of the code  $Q$  of  $\overline{G}$ . Hence, the  $\mathbb{F}_2$ -sum of any two distinct check nodes in  $\overline{G}$  is again a check node in  $\overline{G}$ . We will iteratively modify  $D$  until it becomes primitive by *repeated XORing* of check nodes. The basic operation is the Switch operation in Algorithm 1, which given a variable node  $i \in V$  and distinct check nodes  $j, j' \in C$  such that  $(j, i)$  and  $(i, j')$  are edges in  $D$ , modifies  $D$  by replacing either  $j$  or  $j'$  with the XOR  $j''$  of  $j$  and  $j'$ . A key property of the Switch operation is that it does not increase the indegree or the outdegree of  $i$  and it decreases at least one of them. The Switch operation uses the fact that  $D$  is acyclic.

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**Algorithm 1** Basic Switch operation

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**Switch**  $D$  along path  $j \rightarrow i \rightarrow j'$

Input: variable node  $i \in V$  and check nodes  $j, j' \in C$  such that  $(j, i)$  and  $(i, j')$  are edges in  $D$

- 1: Let  $P = \min\{|w(j, i)|, |w(i, j')|\}$
- 2: Decrease by  $P$  the absolute weights of all the directed edges connected to  $j$  or  $j'$
- 3: Let  $i'$  be the (unique) variable node such that  $(j', i')$  is an edge in  $D$
- 4: Let  $j''$  be the XOR of  $j$  and  $j'$
- 5: Increase by  $P$  the absolute weights the edges  $(j'', i')$  and  $(i'', j'')$ ,  $\forall i'' \neq i' \in N(j'')$
- 6: Remove all zero weight edges.

Figures 1 and 2 illustrate the Switch operation.

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**Claim 4.3 (Switch operation properties)** *Let  $i \in V$  and  $j, j' \in C$  such that  $(j, i)$  and  $(i, j')$  are edges in  $D$ . After switching  $D$  along  $j \rightarrow i \rightarrow j'$ , the followings hold:*

- a)  $D$  is still an acyclic WDG of a hyperflow for  $\gamma$  in  $\overline{G}$ .
- b) For each variable node  $v \in V$ , the total inflow  $F_v^{in}(w)$  to  $v$  and the total outflow  $F_v^{out}(w)$  from  $v$  do not increase.
- c) The indegree of  $i$  and the outdegree of  $i$  do not increase and at least one of them decreases by at least one.

*Proof.* First we note that due to the acyclicity of  $D$ , variable node  $i'$  will not cancel out after XORing  $j$  and  $j'$  in Line 4. Indeed, assume that  $i'$  cancels out, then  $i'$  must be connected to  $j$  (by an edge incoming from  $i'$  since  $j$  already has an edge outgoing to  $i$ ), hence we get the cycle  $j \rightarrow i \rightarrow j' \rightarrow i' \rightarrow j$ .

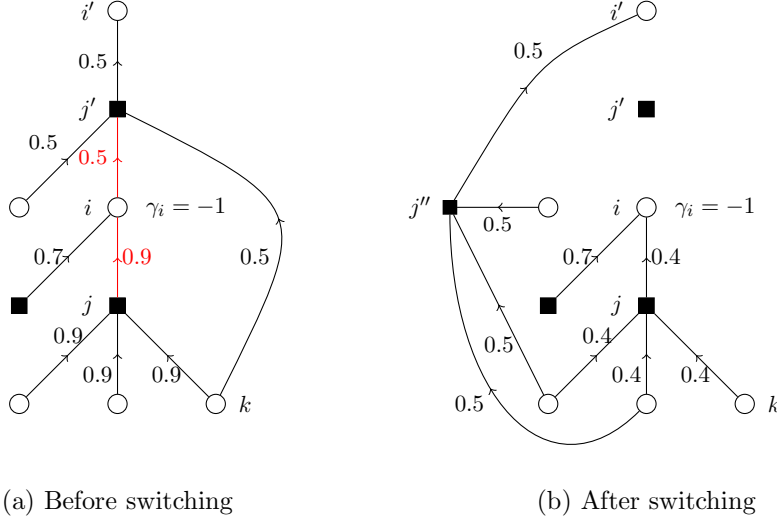


Figure 1: An example of a portion of the WDG  $D$  before and after switching along path  $j \rightarrow i \rightarrow j'$ . This figure illustrates the case when  $|w(i, j')| < |w(j, i)|$ , hence  $P = |w(i, j')|$ .

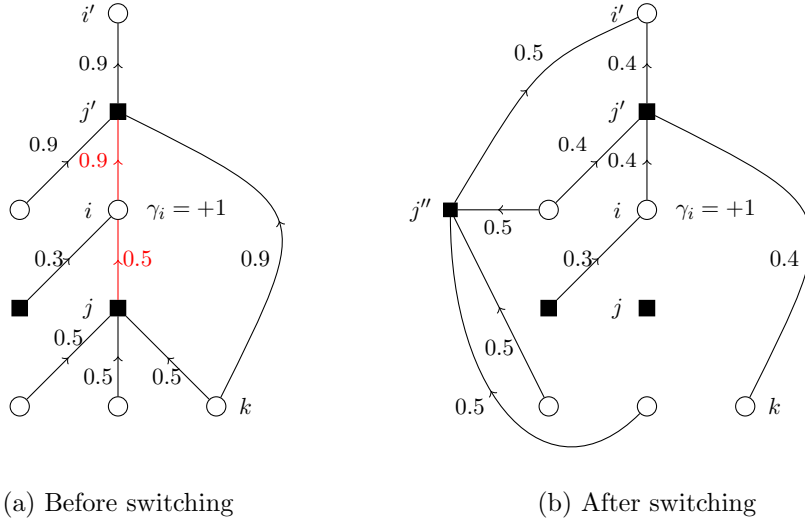


Figure 2: An example of a portion of the WDG  $D$  before and after switching along path  $j \rightarrow i \rightarrow j'$ . This figure illustrates the case when  $|w(i, j')| > |w(j, i)|$ , hence  $P = |w(j, i)|$ .

It is straightforward to verify (b) and (c). Note that the only variable nodes in  $D$  whose inflow or outflow change are those shared by  $j$  and  $j'$  – namely,  $i$  and possibly other nodes  $k$  (see Figures 1 and 2). Both the inflow to  $i$  and the outflow from  $i$  decrease by  $P$ , the outflow from  $k$  decreases by  $P$  and the inflow to  $k$  remains unchanged.

It is also straightforward to verify that the acyclicity of  $D$  and the WDG variable nodes inequalities ((d) in Definition 2.1) are maintained. To complete the proof of (a), we need to show that the hyperflow check nodes inequalities ((c) in Definition 2.1) are maintained. In particular, we have to argue that in Line 5 it is not possible that check node  $j''$  is already present with a

different edge orientation, i.e., with an edge outgoing from  $j''$  to a variable node  $i'' \neq i'$ . Again, this follows from the acyclicity of  $D$ . Assume that right before executing Line 5, there is an edge outgoing from  $j''$  to a variable node  $i'' \neq i'$ . Since variable  $i''$  appears in check  $j''$ , then it appears in either  $j$  or  $j'$ , hence either  $(i'', j)$  or  $(i'', j')$  is an edge in  $D$ . If  $(i'', j)$  is an edge, we get the cycle  $i'' \rightarrow j \rightarrow i \rightarrow j' \rightarrow i' \rightarrow j'' \rightarrow i''$ . If  $(i'', j')$  is an edge, we get the cycle  $i'' \rightarrow j' \rightarrow i' \rightarrow j'' \rightarrow i''$ .  $\blacktriangledown$

Algorithm 2 given below iteratively modifies  $D$  until it becomes primitive by repeated application of the Switch operation. Recall that  $In_D(i)$  is the set of check nodes incident to edges ingoing to  $i$  and  $Out_D(i)$  is the set of check nodes incident to edges outgoing from  $i$ .

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**Algorithm 2** Making the WDG  $D$  primitive

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1: for each variable node  $i \in V$  do
2:   while  $InDegree_D(i) \neq 0$  and  $OutDegree_D(i) \neq 0$  (i.e.,  $In_D(i) \neq \emptyset$  and  $Out_D(i) \neq \emptyset$ ) do
3:     Pick any  $j \in In_D(i)$  and any  $j' \in Out_D(i)$ 
4:     Switch  $D$  along  $j \rightarrow i \rightarrow j'$ 
5:   end while
6: end for
7: for each variable node  $i \in V$  such that  $\gamma_i > 0$  and  $InDegree_D(i) \neq 0$  do
8:   Remove all the edges in  $D$  connected to check nodes in  $In_D(i)$ 
9: end for

```

---

For each  $i \in V$ , Part (c) of Claim 4.3 asserts that the indegree and the outdegree of  $i$  do not increase and at least one of them decreases by at least one, hence the inner while-loop halts in a finite number of steps. Thus at the end of each iteration of the first outer for-loop, variable node  $i$  has either zero indegree or zero outdegree. Part (b) of Claim 4.3 guarantees that once the indegree or the outdegree of a node  $i$  is zero, it remains zero in future iterations of the algorithm.

Consider  $D$  after the end of the first outer for-loop and consider any variable node  $i \in V$ .

If  $\gamma_i \leq 0$ , the indegree of  $i$  must be nonzero due to the WDG variable nodes inequalities. Thus the outdegree of  $i$  must be zero.

If  $\gamma_i > 0$  and the indegree of  $i$  is nonzero, then the outdegree of  $i$  must be zero, hence the outflow from  $i$  is zero. Since  $\gamma_i > 0$  and the outflow from  $i$  is zero, the inflow to  $i$  is unnecessary. The second for-loop performs a final pass to removes this unnecessary inflow by disconnecting the edges of the check nodes in  $In_D(i)$  from  $D$  (thus now both the indegree and the outdegree of  $i$  are zeros).

■

## 5 Impact of redundant checks

In this section we establish Theorem 1.2 and Corollary 1.7 restated below for convenience. The proof of Theorem 1.2 uses the LP excess lemma [BGU14].

**Lemma 5.1 ([BGU14]) (LP Excess Lemma: trading crossover probability with LP excess)** *Let  $H = (V, C, E)$  be a Tanner graph. Let  $0 < \epsilon < \epsilon' < 1$  and  $0 < \delta < 1$  such that  $\epsilon' = \epsilon + (1 - \epsilon)\delta$ . Let  $q_{\epsilon'}$  be the probability that the LP decoder of  $H$  fails on the  $\epsilon'$ -BSC. Consider operating on the  $\epsilon$ -BSC, i.e., choose the error pattern  $x \sim \text{Ber}(\epsilon, n)$ . Then the probability that there exists a dual witness in  $H$  for  $(-1)^x - \frac{\delta}{2}$  is at least  $1 - \frac{2q_{\epsilon'}}{\delta}$ .*

In other words, if we let  $(-1)^{x_i} - \sum_{j \in N(i)} w(i, j)$  be the “LP excess” of  $w$  on variable node  $i$ , then the probability over the  $\epsilon$ -BSC that there exists a dual witness with LP excess greater than  $\frac{\delta}{2}$  on all the variable nodes is at least  $1 - \frac{2q_{\epsilon'}}{\delta}$ .

**Theorem 1.2 (High degree redundant checks do not improve LP threshold)** *Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs such that each check node has degree at most  $d$ , where  $d$  is a constant. Assume that  $\mathcal{G}$  is asymptotically strong. Then:*

- a) *For any small constant  $\delta > 0$ , there exists a sufficiently large constant  $k \geq d$  (dependent on  $\delta$  and independent of  $n$ ) such that  $\xi_{LP}(\overline{\mathcal{G}}^k) \geq \xi_{LP}(\overline{\mathcal{G}}) - \delta$ .*
- b) *If  $k(n)$  is a real valued function of  $n$  such that  $k(n) = \omega(1)$  (i.e.,  $k(n)$  tends to infinity as  $n$  tends to infinity), then  $\xi_{LP}(\overline{\mathcal{G}}^{k(n)}) = \xi_{LP}(\overline{\mathcal{G}})$ .*

**Proof:** Part (b) is an immediate consequence of (a). At a high level, the argument behind (a) is as follows. We will operate  $\overline{\mathcal{G}}$  on the BSC slightly below its LP threshold to guarantee the existence of a dual witness  $w$  with some small but constant LP excess over all variable nodes. Namely, we set the LP excess to  $\frac{\delta}{4}$ . Since  $\overline{\mathcal{G}}$  contains all redundant check nodes, we can assume that  $w$  is *primitive*. We will trim  $w$  by removing all check nodes of degree larger than  $k$ . The trimming process leads to a distorted dual witness  $w^k$ , where the variable nodes inequalities are violated for  $w^k$  over some set of variables which we call *problematic*. Call a variable *risky* if it receives at least  $\frac{\delta}{8}$  flow from the removed check nodes and let  $U$  be the set of risky variables. Thus the risky variables include all the problematic variables. Moreover, all the risky variables are received in error since  $w$  is primitive. Due to the high degree of the removed check nodes and due to the primitivity of  $w$ , the removed checks give the variables in error little flow, namely at most  $\frac{n}{k-1}$ . It follows that the set  $U$  of risky variables is small, namely  $|U| \leq \frac{8n}{\delta(k-1)}$ . Due to the primitivity of  $w$ , the variables in error, and in particular the problematic variables, have no outgoing edges. That is, the outflow from each problematic variable node is zero, hence fixing each problematic variable requires adding a unit flow in the worst case (this conclusion critically depends on the primitivity of  $w$ ). By construction, the nonrisky variables still have  $\frac{\delta}{4} - \frac{\delta}{8} = \frac{\delta}{8}$  LP excess after the trimming process. We will use this remaining excess to fix  $w^k$  by patching a dual witness which turns the remaining small LP excess on the nonrisky variables into a unit flow on each risky variable. The existence of the patch follows from the *asymptotic strength* of  $\mathcal{G}$ .

More formally, let  $\delta > 0$  and assume without loss of generality that  $\xi_{LP}(\overline{\mathcal{G}}) > 0$  and  $\delta < \xi_{LP}(\overline{\mathcal{G}})$  (otherwise, the claim of the theorem is trivial). We will show that there is a sufficiently large constant  $k$  such that  $\xi_{LP}(\overline{\mathcal{G}}^k) \geq \xi_{LP}(\overline{\mathcal{G}}) - \delta$ . Let  $\epsilon = \xi_{LP}(\overline{\mathcal{G}}) - \delta$  and  $\epsilon' = \epsilon + (1 - \epsilon)\frac{\delta}{2}$ , thus  $0 < \epsilon < \epsilon' < \xi_{LP}(\overline{\mathcal{G}})$ . Let  $q_{\epsilon'}(n)$  be the probability of error of the LP decoder of  $\overline{G}_n$  over the  $\epsilon'$ -BSC. Note that  $q_{\epsilon'}(n)$  tends to zero as  $n$  tends to infinity since  $\epsilon' < \xi_{LP}(\overline{\mathcal{G}})$ . By the LP excess lemma (Lemma 5.1), with probability at least  $1 - \frac{4q_{\epsilon'}(n)}{\delta}$ , there exists a dual witness in  $\overline{G}_n$  for  $(-1)^x - \frac{\delta}{4}$ , where  $x \sim \text{Ber}(\epsilon, n)$ . In what follows, consider any  $k$  and  $n$  such that  $d \leq k \leq n$ , consider any  $x \in \{0, 1\}^n$  such that  $(-1)^x - \frac{\delta}{4}$  has a dual witness  $w$  in  $\overline{G}_n$ , say that  $\overline{G}_n = (V, \overline{C}, \overline{E})$  and consider the Tanner graph  $\overline{G}_n^k = (V, \overline{C}^k, \overline{E}^k)$ . We will construct from  $w$  a dual witness for  $(-1)^x$  in  $\overline{G}_n^k$  for sufficiently large  $k$ .

Let  $V_x^+ = \{i \in V : (-1)^{x_i} - \frac{\delta}{4} \geq 0\}$  and  $V_x^- = \{i \in V : (-1)^{x_i} - \frac{\delta}{4} < 0\}$ . Note that since  $0 < \delta < 1$ ,  $V_x^+ = \{i \in V : (-1)^{x_i} = 1\}$  and  $V_x^- = \{i \in V : (-1)^{x_i} = -1\}$ , i.e.,  $V_x^+$  is the set

of variable nodes received correctly and  $V_x^-$  consists of those received in error. Since  $\overline{G_n}$  contains all redundant check nodes, we can assume by Lemma 4.2 that the WDG  $D$  of  $w$  is a *primitive hyperflow*. Since  $D$  is a primitive hyperflow, for each check node  $j$  in  $D$ , all the ingoing edges to  $j$  are from variables in  $V_x^+$  and the only outgoing edge from  $j$  is to some variable in  $V_x^-$ . Let  $L^k$  be the set of check nodes in  $\overline{G_n}$  of degree larger than  $k$ , i.e.,  $L^k = \overline{C} - \overline{C}^k$ . The check nodes in  $L^k$  give the variable nodes in  $V_x^-$  a total flow which is at most  $\frac{|V_x^+|}{k-1} \leq \frac{n}{k-1}$ . Call a variable node in  $V_x^-$  *risky* if it receives at least  $\frac{\delta}{8}$  flow in total from the checks in  $L^k$ . Let  $U$  be the set of risky variable nodes, thus

$$|U| \leq \frac{8n}{\delta(k-1)}.$$

Remove from  $D$  all the check nodes in  $L^k$  and all the associated edges and let  $w^k$  be the resulting weight map  $w^k : \overline{E}^k \rightarrow \mathbb{R}$ . The map  $w^k$  possibly violates the variable nodes inequalities over some variables in  $U$ , but it satisfies the hyperflow check nodes inequalities and hence the dual witness check nodes inequalities over all checks. For each  $i \in V$ , consider the flows at  $i$  associated with  $w$  and  $w^k$ :  $F_i(w) = \sum_j w(i, j)$ ,  $F_i^{out}(w) = \sum_{j \leftarrow i} |w(i, j)|$ ,  $F_i(w^k) = \sum_j w^k(i, j)$ ,  $F_i^{in}(w^k) = \sum_{j \rightarrow i} |w^k(i, j)|$  and  $F_i^{out}(w^k) = \sum_{j \leftarrow i} |w^k(i, j)|$ . Since  $w$  is primitive, none of the variables  $i \in V_x^-$  have outgoing edges in  $\overline{G}$ , thus  $F_i^{out}(w) = 0$  and hence  $F_i^{out}(w^k) = 0$ . Thus for each  $i \in U$ ,  $F_i(w^k) = -F_i^{in}(w^k) \leq 0$ . If  $i \in V_x^- - U$ , we have

$$F_i(w^k) < F_i(w) + \frac{\delta}{8} < -1 - \frac{\delta}{4} + \frac{\delta}{8} = -1 - \frac{\delta}{8}.$$

If  $i \in V_x^+$ , we have  $F_i(w^k) \leq F_i(w) < 1 - \frac{\delta}{4} < 1 - \frac{\delta}{8}$ . Therefore, for each variable  $i \in V$ ,

$$\begin{cases} F_i(w^k) & \leq 0 & \text{if } i \in U \\ F_i(w^k) & < (-1)^{x_i} - \frac{\delta}{8} & \text{otherwise.} \end{cases} \quad (8)$$

To turn  $w^k$  into a dual witness for  $(-1)^x$ , we have to fix the possible violations of variable nodes inequalities over  $U$ . Over  $V - U$ , the variable nodes inequalities are satisfied with  $\frac{\delta}{8}$  excess. We will use this excess to fix the problematic variables in  $U$  by patching to  $w^k$  a dual witness for the asymmetric LLR vector  $\gamma \in \mathbb{R}^V$  given by

$$\gamma_i = \begin{cases} -1 & \text{if } i \in U \\ \frac{\delta}{8} & \text{otherwise,} \end{cases}$$

for all  $i \in V$ .

Since  $\mathcal{G}$  is *asymptotically strong*, there exists a constant  $\alpha_\delta > 0$  (independent of  $n$ ) such that if  $|U| \leq \alpha_\delta n$ , the LP decoder of  $G_n = (V, C, E)$  succeeds on the asymmetric LLR vector  $\gamma$ . Hence, if  $\frac{8}{\delta(k-1)} \leq \alpha_\delta$ , then  $\gamma$  has a dual witness  $v : E \rightarrow \mathbb{R}$  in  $G_n$ . Since  $k \geq d$  (recall that  $d$  is the maximum degree of a check node in  $\mathcal{G}$ ), we can extend  $v$  from  $E$  to  $\overline{E}^k$  by zeros. Let  $v^k : \overline{E}^k \rightarrow \mathbb{R}$  be the resulting weight map, thus

$$\begin{cases} F_i(v^k) & < -1 & \text{if } i \in U \\ F_i(v^k) & < \frac{\delta}{8} & \text{otherwise,} \end{cases} \quad (9)$$

where  $F_i(v^k) = \sum_j v^k(i, j)$ . Since  $U \subset V_x^-$ , it follows from (8) and (9) that  $F_i(w^k) + F_i(v^k) < (-1)^{x_i}$ , for all  $i \in V$ . Noting that the dual witness check nodes inequalities are preserved by superposition, we conclude that  $w^k + v^k$  is the desired dual witness of  $(-1)^x$ .

In summary, for all  $\delta > 0$  such that  $\delta < \xi_{LP}(\overline{\mathcal{G}})$ , there exists a constant  $\alpha_\delta > 0$  such that with  $\epsilon = \xi_{LP}(\overline{\mathcal{G}}) - \delta$ ,  $\epsilon' = \epsilon + (1 - \epsilon)\frac{\delta}{2}$  and  $k = \lceil \frac{8}{\delta\alpha_\delta} \rceil + 1$ , the following holds for all values of  $n$ . Let  $q_{\epsilon'}(n)$  be the probability of error of the LP decoder of  $\overline{G}_n$  over the  $\epsilon'$ -BSC. Then there exists a dual witness in  $\overline{G}_n^k$  for  $(-1)^x$  with probability at least  $1 - \frac{4q_{\epsilon'}(n)}{\delta}$  over the choice of  $x \sim \text{Ber}(\epsilon, n)$ . Since  $\epsilon' < \xi_{LP}(\overline{\mathcal{G}})$ ,  $q_{\epsilon'}(n)$  tends to zero as  $n$  increases. It follows that, for all  $\delta > 0$ , there exists a sufficiently large constant  $k > 0$  dependent on  $\delta$  such that  $\xi_{LP}(\overline{\mathcal{G}}^k) \geq \xi_{LP}(\overline{\mathcal{G}}) - \delta$ . ■

To derive Corollary 1.7 from Theorem 1.2, we need the following classical result.

**Theorem 5.2 ([Fel03]) (Optimality of LP decoding on acyclic graphs)** *Let  $H = (V, C, E)$  be a Tanner graph and  $Q_H$  the associated code. If  $H$  is acyclic, then the fundamental polytope  $P(H)$  of  $H$  is the convex span of the code  $Q_H$ , i.e.,  $\text{conv}(Q_H) = P(H)$ .*

See [VK05] or [BG11] for a proof. It follows from Theorem 5.2 that redundant checks obtained by acyclic sums do not tighten the polytope. A statement similar to Corollary 5.3 appears in [BG11]. We include a short derivation of Corollary 5.3 from Theorem 5.2 for completeness.

**Corollary 5.3 (Acyclic redundant checks do not tighten the polytope)** *Let  $G = (V, C, E)$  be a Tanner graph and  $Q \subset \mathbb{F}_2^n$  the associated code. For each check  $j \in C$ , let  $z_j \in Q^\perp$  be the vector in the dual code associated with  $j$ . Let  $D \subset Q^\perp$  such that each check  $z \in D$  is obtained by an acyclic sum of checks of  $G$ . That is, each  $z \in D$  is of the form  $z = \sum_{j \in S} z_j$ , for some  $S \subset C$  such that the graph induced by  $G$  on  $S$  is acyclic. Consider the Tanner graph  $G' = (V, C \cup D, E')$  resulting from  $G$  by adding all the checks in  $D$ . Then  $P(G) = P(G')$ .*

**Proof:** By definition,  $P(G) = \bigcap_{j \in C} \text{conv}(Q_j)$  and  $P(G') = \bigcap_{z \in C \cup D} \text{conv}(Q_z)$ . Consider any check  $z \in D$ . It is enough to argue that  $P(G) \subset \text{conv}(Q_z)$ . Let  $S \subset C$  such that  $z = \sum_{j \in S} z_j$  and the graph  $G_S = (V_S, S, E_S)$  induced by  $G$  on  $S$  is acyclic. By Theorem 5.2,  $P(G_S) = \text{conv}(Q_{G_S})$ . Extending the polytopes from  $\mathbb{R}^{V_S}$  to  $\mathbb{R}^V$ , we get  $\bigcap_{j \in S} \text{conv}(Q_j) = \text{conv}(Q^S)$ , where  $Q^S$  is the supercode of  $Q$  consisting of all the vectors in  $\mathbb{F}_2^n$  satisfying all the checks in  $S$ . Since  $z$  is a linear combinations of checks in  $S$ , we have  $Q^S \subset Q_z$ , hence  $\text{conv}(Q^S) \subset \text{conv}(Q_z)$ . Therefore

$$P(G) = \bigcap_{j \in C} \text{conv}(Q_j) \subset \bigcap_{j \in S} \text{conv}(Q_j) = \text{conv}(Q^S) \subset \text{conv}(Q_z).$$

■

Finally, we conclude Corollary 1.7 from Theorem 1.2 and Corollary 5.3.

**Corollary 1.7 (Redundant checks do not improve LP threshold)** *Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs of bounded check degree. If  $\mathcal{G}$  is asymptotically strong and rigid, then  $\xi_{LP}(\overline{\mathcal{G}}) = \xi_{LP}(\mathcal{G})$ .*

**Proof:** Say that  $G_n = (V_n, C_n, E_n)$  and  $\overline{G}_n = (V_n, \overline{C}_n, \overline{E}_n)$ . Since  $\mathcal{G}$  is rigid,  $\Delta(G_n) = \omega(1)$ . Let  $k(n) := \Delta(G_n) - 1$  and assume that  $n$  is large enough so that  $k(n)$  is at least the maximum degree of a check node of  $\mathcal{G}$ . By the definition of  $k(n)$ , all redundant checks in  $\overline{C}_n$  of degree at most  $k(n)$  are obtained by acyclic sums of checks in  $C_n$ . By Corollary 5.3,  $P(G_n) = P(\overline{G}_n^{k(n)})$ , hence  $\xi_{LP}(\mathcal{G}) = \xi_{LP}(\overline{\mathcal{G}}^k)$ . On the other hand, by Theorem 1.2,  $\xi_{LP}(\overline{\mathcal{G}}^k) = \xi_{LP}(\overline{\mathcal{G}})$  since  $k(n) = \omega(1)$  and  $\mathcal{G}$  is asymptotically strong. It follows that  $\xi_{LP}(\mathcal{G}) = \xi_{LP}(\overline{\mathcal{G}})$ . ■



## 6 Expansion and asymptotic strength

In this section we prove Theorem 1.3 restated below for convenience. The proof uses the notion of a narrow dual witness defined below.

**Definition 6.1 (Narrow dual witness)** *Let  $G = (V, C, E)$  be a Tanner graph,  $y \in \{0, 1\}^n$  an error vector and  $w : E \rightarrow \mathbb{R}$  a dual witness for  $(-1)^y$  in  $G$ . We call  $w$  a narrow dual witness for  $(-1)^y$  if all the edges not incident to  $N(U)$  have zero weights, where  $U = \{i \in V : y_i = 1\}$  is the set of variables in error (i.e., if an edge is not incident to a check node incident to a variable in error, then it has zero weight).*

A key property of a narrow dual witness is that the flow at the correct variable nodes far from  $U$  by more than 2 edges is zero.

Recall that a Tanner graph  $G = (V, C, E)$  is called an  $(\varepsilon n, \kappa)$ -expander if for each subset  $S \subset V$  of variable nodes of size at most  $\varepsilon n$ , we have  $|N(S)| \geq \kappa|S|$ .

Feldman et al. [FMS+07] argued that the LP decoder of graphs with good expansion corrects a positive fraction of errors. Although not explicitly stated, the dual witness constructed in their proof is actually narrow. Their argument was later simplified by Viderman [Vid13] who also improved the expansion requirement.

**Lemma 6.2 (Implicit in [Vid13]) (Expansion implies the existence of a narrow dual witness)** *Let  $d_v > 0, \varepsilon > 0$  and  $\delta > \frac{2}{3}$  be constants such that  $d_v$  is an integer and  $\delta d_v$  is an integer. Let  $G = (V, C, E)$  be a Tanner graph with regular variable degree  $d_v$  and assume that  $G$  is an  $(\varepsilon n, \delta d_v)$ -expander. Then  $(-1)^y$  has a narrow dual witness in  $G$ , for each error vector  $y \in \{0, 1\}^n$  of weight at most  $\frac{3\delta-2}{2\delta-1}(\varepsilon n - 1)$ .*

**Theorem 1.3 (Expansion implies asymptotic strength)** *Let  $d_v > 0, \varepsilon > 0$  and  $\delta > \frac{2}{3}$  be constants such that  $d_v$  is an integer and  $\delta d_v$  is an integer. Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs with regular variable degree  $d_v$  and bounded check degree. If  $G_n$  is an  $(\varepsilon n, \delta d_v)$ -expander for each  $n$ , then  $\mathcal{G}$  is asymptotically strong.*

**Proof:** The proof is based on successive superpositions of narrow dual witnesses obtained from Lemma 6.2 to amplify the flow at the variable nodes in errors. The fact they are narrow is essential for superposing them without violating the variable nodes constraints at the correct variables.

Consider any constant  $\beta > 0$  and let  $B = \lceil \frac{1}{\beta} \rceil$ . It is enough to find a constant  $\alpha > 0$  and construct, for each  $n$  and each  $U \subset V = \{1, \dots, n\}$  of size most  $\alpha n$ , a dual witness  $w$  in  $G_n = (V, C, E)$  for the asymmetric LLR vector  $\gamma \in \mathbb{R}^V$  given by

$$\gamma_i = \begin{cases} -B & \text{if } i \in U \\ 1 & \text{otherwise,} \end{cases}$$

for all  $i \in V$ . Since  $B \geq \frac{1}{\beta}$ , the scaled version  $\frac{1}{B}w$  of  $w$  is the desired dual witness for  $\gamma(y, \beta)$  (as given in Definition 1.1), where  $y \in \{0, 1\}^n$  is the indicator vector of  $U$ .

If  $S \subset V$  is a set of variable nodes and  $t \geq 0$  is an integer, let  $N_{var}(S; t)$  be the set of variable nodes at distance at most  $2t$  from  $S$ . Thus  $N_{var}(S; 0) = S$  and  $N_{var}(S; 1)$  is the set of variables connected to check nodes connected to  $S$ .

Let  $\alpha > 0$  be a sufficiently small constant such that for each  $U \subset V$  of size at most  $\alpha n$ , we have

$$|N_{var}(U; B-1)| \leq \frac{3\delta-2}{2\delta-1}(\varepsilon n - 1), \quad (10)$$

for sufficiently large  $n$  (the explicit value of  $\alpha$  is at the end of the proof). Assume that  $|U| \leq \alpha n$  and let  $U^t = N_{var}(U; t)$ , for  $t = 0, \dots, B-1$ . In what follows, consider any  $t \in \{0, \dots, B-1\}$ . Since  $|U^t| \leq \frac{3\delta-2}{2\delta-1}(\varepsilon n - 1)$ , Lemma 6.2 guarantees that  $(-1)^{y^t}$  has a narrow dual witness  $w^t : E \rightarrow \mathbb{R}$  in  $G$ , where  $y^t \in \{0, 1\}^n$  is the indicator vector of  $U^t$ , i.e.,  $y_i^t = 1$  iff  $i \in U^t$ . The fact that  $w^t$  is narrow means all the edges not incident to  $N(U^t)$  have zero weights, thus the flow at the variable nodes outside  $U^{t+1}$  is zero. That is,  $F_i(w^t) = 0$  for each  $i \in V - U^{t+1}$ , where  $F_i(w^t) = \sum_j w^t(i, j)$  is the flow with respect to  $w^t$  at variable node  $i$ . Let  $w = \sum_{t=0}^{B-1} w^t$ . We will argue that  $w$  is the desired dual witness for  $\gamma$ .

First, note that superposing dual witnesses does not violate the dual witness check nodes inequalities ((b) in Definition 2.1). Thus, we only have to worry about the variable nodes inequalities ((a) in Definition 2.1). Consider the flow at the variable nodes with respect to  $w$ :  $F_i(w) = \sum_{t=0}^{B-1} F_i(w^t)$ , for all  $i \in V$ . We have to show that

$$\begin{cases} F_i(w) < -B & \text{if } i \in U^0 = U \\ F_i(w) < 1 & \text{otherwise.} \end{cases} \quad (11)$$

Since each  $w^t$  is a narrow dual witness for  $(-1)^{y^t}$ , we have

$$\begin{cases} F_i(w^t) < -1 & \text{if } i \in U^t \\ F_i(w^t) < 1 & \text{if } i \in U^{t+1} - U^t \\ F_i(w^t) = 0 & \text{if } i \in V - U^{t+1}. \end{cases}$$

Summing over  $t = 0, \dots, B-1$  and using the fact that  $U^0 \subset U^1 \subset U^2 \subset \dots \subset U^B$ , we obtain

$$\begin{cases} F_i(w) < -B & \text{if } i \in U^0 \\ F_i(w) < -(B-2) & \text{if } i \in U^1 - U^0 \\ F_i(w) < -(B-3) & \text{if } i \in U^2 - U^1 \\ F_i(w) < -(B-4) & \text{if } i \in U^3 - U^2 \\ \dots & \\ F_i(w) < -2 & \text{if } i \in U^{B-3} - U^{B-4} \\ F_i(w) < -1 & \text{if } i \in U^{B-2} - U^{B-3} \\ F_i(w) < 0 & \text{if } i \in U^{B-1} - U^{B-2} \\ F_i(w) < 1 & \text{if } i \in U^B - U^{B-1} \\ F_i(w) = 0 & \text{if } i \in V - U^B, \end{cases}$$

and hence (11) follows.

Finally, note that if  $d_c$  be the maximum check degree of check node in  $G_n$  for all  $n$ , then for all  $t \geq 0$ ,

$$|N_{var}(U; t)| \leq \sum_{i=0}^t (d_v(d_c - 1))^i |U| = \frac{(d_v(d_c - 1))^{t+1} - 1}{d_v(d_c - 1) - 1} |U|.$$

Thus condition (10) is satisfied if

$$\frac{(d_v(d_c - 1))^B - 1}{d_v(d_c - 1) - 1} \alpha n \leq \frac{3\delta-2}{2\delta-1}(\varepsilon n - 1),$$

which holds for  $n$  sufficiently large with

$$\alpha = \frac{(3\delta - 2)(d_v(d_c - 1) - 1)}{(4\delta - 2)((d_v(d_c - 1))^{\lceil \frac{1}{\beta} \rceil} - 1)} \varepsilon.$$

■

## 7 Nondegeneracy of random graphs

In this section we prove Lemmas 1.9 and 1.10 restated below for convenience.

**Lemma 1.9 (Rigidity versus girth and nondegeneracy)** *Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs of bounded check degree. If  $\mathcal{G}$  is rigid, then  $\text{girth}(G_n) = \omega(1)$ . On the other hand, if  $\text{girth}(G_n) = \Theta(\log n)$  and the minimum check degree of  $\mathcal{G}$  is at least 3 (i.e., for all  $n$ , each check node of  $G_n$  has degree at least 3), then the following are equivalent:*

- i) (Rigidity)  $\mathcal{G}$  is rigid
- ii) (Nondegeneracy) For each constant  $c > 0$ ,  $G_n$  is  $(c \log n, \omega(1))$ -nondegenerate. That is, for each constant  $c > 0$ , the minimum weight of a sum of at least  $c \log n$  checks nodes tends to infinity as  $n$  increases.

**Proof:** First we show that if  $\mathcal{G}$  is rigid, then  $\text{girth}(G_n) = \omega(1)$ . If  $G_n$  has a cycle of  $O(1)$  length, then the weight of the sum of the check nodes on this cycle is  $O(1)$  since  $\mathcal{G}$  has bounded check degree, which contradicts the rigidity of  $\mathcal{G}$ .

Assume in what follows that:

- a)  $\text{girth}(G_n) = \Theta(\log n)$  and let  $\alpha > 0$  be a constant such that  $\text{girth}(G_n) \geq \alpha \log n$  for sufficiently large  $n$ .
- b) Each check node of  $G_n$  has degree at least 3.

Say that  $G_n = (V_n, C_n, E_n)$ , let  $Q_n$  be the code associated with  $G_n$  and let  $z_j \in Q_n^\perp$  be the vector in the dual code associated with check  $j \in C_n$ . We will use (a) to show that (ii) implies (i) and (b) to show that (i) implies (ii).

Assume that (ii) holds. To verify (i), let  $z = \sum_{j \in S} z_j$  for some subset  $S \subset C_n$  such that the graph induced by  $G_n$  on  $S$  contains a cycle. Thus  $|S| \geq \frac{1}{2} \text{girth}(G) \geq \frac{\alpha}{2} \log n$ . Since  $G_n$  is  $(\frac{\alpha}{2} \log n, \omega(1))$ -nondegenerate, we get  $\text{weight}(z) = \omega(1)$ , hence  $\mathcal{G}$  is rigid.

Finally, assume that  $\mathcal{G}$  is rigid and let  $c > 0$ . To verify that (ii) holds, we use (b). Let  $z = \sum_{j \in S} z_j$  for some subset  $S \subset C_n$  of size at least  $c \log n$ . We will argue that  $\text{weight}(z) = \omega(1)$  by considering two cases depending on whether or not the graph  $G_S$  induced by  $G_n$  on  $S$  is acyclic.

Case 1: Assume that  $G_S$  is acyclic. Since each check node in  $S$  has degree at least 3, the number of leaves in the forest  $G_S$  is at least  $|S| + 2$  (in general, if  $F$  is a forest and  $s$  is the number of internal nodes of  $F$  of degree at least 3, then the number of leaves of  $F$  is at least  $s + 2$  assuming that  $s \geq 1$ ). Since each of those leaves must be a variable node, we get  $\text{weight}(z) \geq |S| + 2 = \Omega(\log n)$ .

Case 2: If  $G_S$  contains a cycle, then  $\text{weight}(z) = \omega(1)$  since  $\mathcal{G}$  is rigid. ■

**Definition 7.1 ([Cal97]) (Calkin's threshold)** If  $d \geq 3$  is an integer, define the threshold  $0 < \beta_d < 1$  as follows. Consider the function

$$f_d(\alpha, \beta) = -1 + H(\alpha) + \beta \log_2(1 + (1 - 2\alpha)^d),$$

where  $H(\alpha) = -\alpha \log_2 \alpha - (1 - \alpha) \log_2(1 - \alpha)$  is the binary entropy function. Let

$$\beta_d := \sup\{\beta^* : f_d(\alpha, \beta) < 0 \text{ for all } 0 < \alpha < 1/2 \text{ and all } 0 < \beta < \beta^*\}.$$

Equivalently,  $\beta_d$  is the unique  $0 < \beta_d < 1$  such that there exists  $0 < \alpha_d < 1/2$  such that  $(\alpha_d, \beta_d)$  is a root of the system of equations

$$\begin{cases} f_d(\alpha, \beta) &= 0 \\ \frac{\partial}{\partial \alpha} f_d(\alpha, \beta) &= 0. \end{cases}$$

For instance,  $\beta_3 \sim 0.8895$ ,  $\beta_4 \sim 0.967$  and  $\beta_5 \sim 0.989$ . As  $d$  increases,  $\beta_d$  approaches 1. In general, Calkin shows that  $\beta_d = (1 - \frac{e^{-d}}{\ln 2})(1 \pm o(1))$ . Calkin established the following.

**Lemma 7.2 ([Cal97]) (Random row-regular matrices have full row rank)** Let  $d \geq 3$  be an integer. Consider a random  $m \times n$  matrix  $M \in \mathbb{F}_2^{m \times n}$  constructed by independently choosing each of the  $m$  rows of  $M$  uniformly from the set of vectors in  $\mathbb{F}_2^n$  of weight  $d$ . If  $m < \beta_d n$ , then the probability that the rows of  $M$  are linearly dependent goes to zero as  $n$  tends to infinity.

Note that full row rank corresponds to  $(1, 0)$ -nondegeneracy.

**Lemma 1.10 (Random check-regular graphs are nondegenerate)** Let  $d, m$  and  $n$  be integers such that  $d \geq 3$  and  $1 \leq m < \beta_d n$ . Consider a random  $m \times n$  matrix  $M \in \mathbb{F}_2^{m \times n}$  constructed by independently choosing each of the  $m$  rows of  $M$  uniformly from the set of vectors in  $\mathbb{F}_2^n$  of weight  $d$ . Then for any constant  $c > 0$  and any function  $k(n)$  of  $n$  such that  $k(n) = o(\log \log n)$ ,  $M$  is  $(c \log n, k(n))$ -nondegenerate with high probability. That is, the probability that there are at least  $c \log n$  rows of  $M$  whose  $\mathbb{F}_2$ -sum has weight less than or equal to  $k(n)$  goes to zero as  $n$  tends to infinity.

## 7.1 Proof of Lemma 1.10

The proof follows the argument Calkin [Cal97] used to establish Lemma 7.2. Let  $B_d$  be the set of vectors in  $\mathbb{F}_2^n$  of weight  $d$ . Let  $g = \lceil c \log n \rceil$ ,  $k = k(n)$ , and  $P$  be the probability that there are at least  $g$  rows of  $M$  whose  $\mathbb{F}_2$ -sum has weight less than or equal to  $k$ . Thus

$$P \leq \sum_{t=g}^m \binom{m}{t} \sum_{p=0}^k a_p^{(t)}, \quad (12)$$

where  $a_p^{(t)}$  is the probability that the weight of the sum of  $t$  random vectors chosen uniformly and independently from  $B_d$  is  $p$ .

Consider the random walk on  $\mathbb{F}_2^n$  which starts from 0 and moves by adding random elements from  $B_d$ . The transition probability matrix of the underlying Markov chain is the  $(n+1) \times (n+1)$  matrix  $A = (a_{pq})_{p,q \in \{0, \dots, n\}}$ , where  $a_{pq}$  is defined as follows. Fix any vector  $y_q \in \mathbb{F}_2^n$  of weight  $q$ .

Then  $a_{pq}$  is the probability that the weight of  $x + y_q$  is  $p$  over the uniformly random choice of  $x$  from  $B_d$ . The entries of  $A$  are given by

$$a_{pq} = \frac{\binom{\frac{q}{2}}{\frac{q+d-p}{2}} \binom{\frac{n-q}{2}}{\frac{d-q+p}{2}}}{\binom{n}{d}}$$

if  $q + d - p$  is even. Otherwise,  $a_{pq} = 0$ . In terms of  $A$ ,  $a_p^{(t)} = a_{p0}^{(t)}$ , where  $a_{p0}^{(t)}$  is the  $(p, 0)$ 'th entry of the matrix  $A^t$ .

The following lemma due to Calkin gives the eigenvalues and the eigenvectors of  $A$  in terms of Krawtchouk Polynomials.

**Lemma 7.3** ([Cal97]) *a) The eigenvalues of  $A$  are*

$$\lambda_i = \frac{1}{\binom{n}{d}} \sum_s (-1)^s \binom{i}{s} \binom{n-i}{d-s} \quad \text{for } i = 0, \dots, n.$$

*The eigenvector corresponding to  $\lambda_i$  is the  $n \times 1$  vector  $e_i$  whose entries are given by*

$$e_{ij} = \sum_s (-1)^s \binom{i}{s} \binom{n-i}{j-s} \quad \text{for } j = 0, \dots, n.$$

*Moreover,  $A$  is decomposable as  $A = U\Lambda U^{-1}$ , where  $\Lambda = \text{diag}(\lambda_i)_{i=1}^n$ ,  $U$  is the matrix whose columns are  $e_0, \dots, e_n$  and  $U^{-1} = 2^{-n}U$ .*

*b) If  $i > \frac{n}{2}$ , then  $\lambda_i = (-1)^d \lambda_{n-i}$ .*

We have

$$a_p^{(t)} = a_{p0}^{(t)} = 2^{-n} \sum_i e_{ip} \lambda_i^t e_{0i} \leq 2^{-n} \binom{n}{p} \sum_i \binom{n}{i} |\lambda_i|^t,$$

since  $e_{0i} = \binom{n}{i}$  and  $|e_{ip}| \leq \binom{n}{p}$ . It follows from (12), that

$$\begin{aligned} P &\leq 2^{-n} \sum_{p=0}^k \binom{n}{p} \sum_{i=0}^n \binom{n}{i} \sum_{t=g}^m \binom{m}{t} |\lambda_i|^t \\ &\leq 2(n+1)^k \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{-n} \binom{n}{i} \sum_{t=g}^m \binom{m}{t} |\lambda_i|^t, \end{aligned} \tag{13}$$

where the second inequality follows from Part (b) of Lemma 7.3 and the bound  $\sum_{p=0}^k \binom{n}{p} \leq (n+1)^k$ . Instead of (13), Calkin obtains the bound:

$$2 \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{-n} \binom{n}{i} \sum_{t=1}^m \binom{m}{t} \lambda_i^t. \tag{14}$$

The key differences between (13) and (14) are that (14) starts from  $t = 1$  instead of  $t = g$  and (13) has the extra  $(n+1)^k$  term (the fact that the absolute values of the eigenvalues appear in (13) instead of their actual values is of minor significance). We will show that  $P \leq 2^{-\Theta(n^{1/7}) + \frac{2(n+1)^k m}{g^g}}$ , hence  $P = o(1)$  for  $g = \Theta(\log n)$  and  $k = o(\log \log n)$ .

To estimate  $P$ , we will use the following bounds on the eigenvalues established by Calkin.

**Lemma 7.4** ([Cal97]) a)  $|\lambda_i| \leq 1$  for all  $0 \leq i \leq n$

b) If  $cn \leq i \leq \frac{n}{2}$  for some constant  $c > 0$ , then

$$\lambda_i = \left(1 - \frac{2i}{n}\right)^d - \frac{4\binom{d}{2}}{n} \left(1 - \frac{2i}{n}\right)^{d-2} \frac{i}{n} \left(1 - \frac{i}{n}\right) + O\left(\frac{d^3}{n^2}\right).$$

c) If  $\frac{n}{2} - n^{4/7} \leq i \leq \frac{n}{2}$ , then  $|\lambda_i| = o\left(\frac{1}{n}\right)$ .

Let

$$P_i = 2(n+1)^k 2^{-n} \binom{n}{i} \sum_{t=g}^m \binom{m}{t} |\lambda_i|^t.$$

Thus  $P \leq \sum_{i=0}^{\lfloor n/2 \rfloor} P_i$ . We divide the summation on  $i$  as in the argument of Calkin into three regions:  $0 \leq i \leq \epsilon n$ ,  $\epsilon n < i \leq n - n^{4/7}$  and  $\frac{n}{2} - n^{4/7} < i \leq \frac{n}{2}$ , where  $\epsilon > 0$  is a sufficiently small constant. We will use the condition  $m < \beta_d n$  in second region and fact that  $t$  starts from  $g$  in the third region.

*Region 1:*  $0 \leq i \leq \epsilon n$ . Using the bound  $|\lambda_i| \leq 1$  and ignoring the lower bound  $g$  on  $t$ , we get

$$\begin{aligned} P_i &\leq 2(n+1)^k 2^{-n} \binom{n}{i} \sum_{t=0}^m \binom{m}{t} |\lambda_i|^t = 2(n+1)^k 2^{-n} \binom{n}{i} (1 + |\lambda_i|)^m \\ &\leq 2(n+1)^k \binom{n}{i} 2^{-(n-m)} \leq 2(n+1)^k 2^{-n(1-m/n-H((\epsilon))+O(\log n))} = 2^{-\Theta(n)}, \end{aligned}$$

for sufficiently small  $\epsilon > 0$ , since  $m < \beta_d n$ ,  $\beta_d < 1$  and  $k = o\left(\frac{n}{\log n}\right)$ . Hence

$$P^{(1)} := \sum_{0 \leq i \leq \epsilon n} P_i \leq 2^{-\Theta(n)}.$$

*Region 3:*  $\frac{n}{2} - n^{4/7} < i \leq \frac{n}{2}$ . Here we use the bound  $\lambda_i = o\left(\frac{1}{n}\right)$  in Part (c) of Lemma 7.4 and the bound  $\binom{m}{t} \leq \left(\frac{em}{t}\right)^t$ :

$$\begin{aligned} P_i &\leq 2(n+1)^k 2^{-n} \binom{n}{i} \sum_{t=g}^m \left(\frac{em}{t} |\lambda_i|\right)^t \\ &\leq 2(n+1)^k 2^{-n} \binom{n}{i} \sum_{t=g}^m \frac{1}{t^t} \\ &\leq 2(n+1)^k 2^{-n} \binom{n}{i} \frac{m}{g^g}, \end{aligned}$$

where the second equality holds for sufficiently large  $n$ . Hence

$$P^{(3)} := \sum_{\frac{n}{2} - n^{4/7} < i \leq \frac{n}{2}} P_i \leq \frac{2(n+1)^k m}{g^g}.$$

*Region 2:*  $\epsilon n < i \leq \frac{n}{2} - n^{4/7}$ . As in the first region,

$$P_i \leq 2(n+1)^k 2^{-n} \binom{n}{i} \sum_{t=0}^m \binom{m}{t} |\lambda_i|^t = 2(n+1)^k 2^{-n} \binom{n}{i} (1 + |\lambda_i|)^m$$

Now, we use the bound on  $\lambda_i$  in Part (b) of Lemma 7.4 which implies that

$$|\gamma_i| \leq \left(1 - \frac{2i}{n}\right)^d + O\left(\frac{d^3}{n^2}\right).$$

Thus

$$(1 + |\lambda_i|)^m \leq \left(1 + \left(1 - \frac{2i}{n}\right)^d + O\left(\frac{d^3}{n^2}\right)\right)^m = \left(1 + \left(1 - \frac{2i}{n}\right)^d\right)^m (1 + o(1)).$$

For the binomial coefficients, we use the bound  $\binom{n}{i} \leq \frac{e}{2\pi\sqrt{\epsilon(1-\epsilon)n}} 2^{nH(\frac{i}{n})}$  which holds for  $\epsilon n \leq i \leq n - \epsilon n$  and follows from Stirling's approximation. It follows that

$$P_i \leq \delta n^{k-\frac{1}{2}} 2^{-n(1-H(\frac{i}{n}))} \left(1 + \left(1 - \frac{2i}{n}\right)^d\right)^m,$$

for some absolute constant  $\delta > 0$  and sufficiently large  $n$ . Therefore

$$\begin{aligned} P^{(2)} &:= \sum_{\epsilon n < i \leq \frac{n}{2} - n^{4/7}} P_i \\ &\leq \delta n^{k-\frac{1}{2}} \sum_{\epsilon n < i \leq \frac{n}{2} - n^{4/7}} 2^{-n(1-H(\frac{i}{n}))} \left(1 + \left(1 - \frac{2i}{n}\right)^d\right)^m \\ &= \delta n^{k-\frac{1}{2}} \sum_{\epsilon n < i \leq \frac{n}{2} - n^{4/7}} 2^{nf_d(\frac{i}{n}, \frac{m}{n})}. \end{aligned}$$

By the definition of  $\beta_d$ , we have  $f_d(\frac{i}{n}, \frac{m}{n}) < 0$  for all  $\epsilon n < i \leq \frac{n}{2} - n^{4/7}$  since  $\frac{m}{n} < \beta_d$ . Moreover, since  $f_d(\frac{1}{2}, \beta) = 0$  for each  $\beta$ , the maximum of  $f_d(\frac{i}{n}, \frac{m}{n})$  over  $\epsilon n < i \leq \frac{n}{2} - n^{4/7}$  occurs at  $i = \lfloor \frac{n}{2} - n^{4/7} \rfloor$ . It follows that

$$P^{(2)} \leq \delta n^{k+\frac{1}{2}} 2^{nf_d(\frac{\lfloor \frac{n}{2} - n^{4/7} \rfloor}{n}, \frac{m}{n})}.$$

For  $|\alpha - \frac{1}{2}| = o(1)$  and  $\beta > 0$ ,  $f_d(\alpha, \beta) = -1 + H(\alpha) + \beta \log_2(1 + (1 - 2\alpha)^d) = -\Theta((\alpha - \frac{1}{2})^2)$  since  $H(\alpha) = 1 - \Theta((\alpha - \frac{1}{2})^2)$  and  $d \geq 3$ . It follows that  $P^{(2)} \leq \delta n^{k+\frac{1}{2}} 2^{-\Theta(n^{1/7})} = 2^{-\Theta(n^{1/7})}$  for  $k = o\left(\frac{n^{1/7}}{\log n}\right)$ .

Combining the above three cases, we get

$$P \leq P^{(1)} + P^{(2)} + P^{(3)} \leq 2^{-\Theta(n^{1/7})} + \frac{2(n+1)^k m}{g^g}$$

if  $k = o\left(\frac{n^{1/7}}{\log n}\right)$ . Recall that  $g = \Theta(\log n)$ . It follows that  $P = o(1)$  if  $k = o(\log \log n)$ .

## 8 Pseudocodewords interpretation of asymptotic strength

In the section we give an interpretation of the notion of asymptotic strength in terms of the fractional spectrum of pseudocodewords. Then we compare with the related notions of minimum BSC-pseudoweight [VK05], fractional distance and maximum-fractional distance [Fel03, FWK05].

If  $G = (V, C, E)$  is a Tanner graph, let  $\text{Ext}(G)$  be the set of extreme points of  $P(G)$ . The codewords of  $Q$  are the integral vertices of  $P(G)$ , i.e.,  $\text{Ext}(G) \cap \{0, 1\}^n = Q$ . The elements of  $\text{Ext}(G)$  are called *pseudocodewords* (see [Fel03, KV03, FWK05, VK05]).

In terms of pseudocodewords, the notion of asymptotic strength translates as follows.

**Theorem 8.1 (Pseudocodewords and asymptotic strength)** *Let  $\mathcal{G} = \{G_n\}_n$  be an infinite family of Tanner graphs. Then  $\mathcal{G}$  is asymptotically strong iff for each (small) constant  $\theta > 0$ , there exists a constant  $\alpha > 0$  such that for each  $n$  and each nonzero pseudocodeword  $x \in \text{Ext}(G_n)$ , the sum of the largest  $\lfloor \alpha n \rfloor$  coordinates of  $x$  is less than  $\theta \sum_i x_i$ . That is, to attain a positive fraction of  $\sum_i x_i$ , we need a least linear number of coordinates of  $x$ .*

**Proof:** By the definition of the LP decoder, the following are equivalent for any LLR vector  $\gamma \in \mathbb{R}^n$ :

- i) The LP decoder of  $G_n = (V_n, C_m, E_n)$  succeeds on  $\gamma$  under the all-zeros codeword assumption
- ii)  $\langle x, \gamma \rangle > 0$  for each nonzero pseudocodeword  $x \in \text{Ext}(G)$ .

By the equivalence between (i) and (ii),  $\mathcal{G}$  is asymptotically strong iff for each constant  $\beta > 0$ , there exists a constant  $\alpha > 0$  such that for each  $n$  and each error vector  $y \in \{0, 1\}^n$  of weight at most  $\alpha n$ , we have  $\langle x, \gamma(y, \beta) \rangle > 0$  for each nonzero pseudocodeword  $x \in \text{Ext}(G_n)$ , where  $\gamma(y, \beta) : V_n \rightarrow \mathbb{R}$  is the asymmetric LLR vector given by

$$\gamma_i(y, \beta) = \begin{cases} -1 & \text{if } y_i = 1 \\ \beta & \text{if } y_i = 0. \end{cases}$$

Let  $U = \{i : y_i = 1\}$ , thus

$$\langle x, \gamma(y, \beta) \rangle = - \sum_{i \in U} x_i + \beta \sum_{i \in U^c} x_i = -(1 + \beta) \sum_{i \in U} x_i + \beta \sum_i x_i.$$

Hence  $\langle x, \gamma(y, \beta) \rangle > 0$  is equivalent to  $\sum_{i \in U} x_i < \frac{\beta}{1+\beta} \sum_i x_i$ . The theorem then follows by setting  $\theta = \frac{\beta}{1+\beta}$ . ■

Note that if  $x$  is integral, i.e.,  $x \in \{0, 1\}^n$  is a codeword, then the above condition is equivalent to  $\text{weight}(x) = \Theta(n)$ . If  $x$  is not integral, the above condition says that the fractional weights spectrum is not “too unbalanced” in the sense that we need at least a linear number of coordinates of  $x$  to attain a positive fraction of  $\sum_i x_i$ .

In the setup of Theorem 8.1, the minimum BSC-pseudoweight [VK05]  $w_p^{BSC}(G_n)$  corresponds to  $\theta = \frac{1}{2}$ . Namely,  $w_p^{BSC}(G_n) = 2a^*$ , where  $a^*$  is the maximum value of  $a$  such that the sum of the largest  $a$  coordinates of  $x$  is less than  $\frac{1}{2} \sum_i x_i$  for all nonzero  $x \in \text{Ext}(G_n)$ . The 2 multiplicative factor ensures that the largest number of errors the LP decoder can handle over the BSC is  $w_p^{BSC}(G_n)/2$ . Thus, for integral codewords, the BSC-pseudoweight coincides with the Hamming weight. The asymptotic strength property implies that  $w_p^{BSC}(G_n) = \Theta(n)$ . It is not clear if the converse holds; the asymptotic strength requirement seems stronger since it is in terms of all  $\theta > 0$  and not only  $\theta = \frac{1}{2}$ . We leave the problem of whether or not it is strictly stronger open.

The fractional distance of  $G$  is the minimum  $L_1$ -norm of a nonzero pseudocodeword [Fel03, FWK05]. Unlike the minimum BSC-pseudoweight, the fractional distance is always sublinear for regular bounded-degree Tanner graphs [KV03, VK05]. The same holds for the maximum-fractional distance which is defined as the minimum  $L_1$ -norm/ $L_\infty$ -norm of a nonzero pseudocodeword [Fel03, FWK05].



## 9 Decoding with help bits

In this section we highlight a general property of asymptotically strong Tanner graphs. We argue that for such graphs, allowing a sublinear number of “help bits” does not improve the LP threshold. This result, although a negative statement, has potential constructive applications as it weakens the dual witness requirement for LP decoding success. We also derive a converse of the LP excess lemma.

**Definition 9.1 (LP decoder with help)** *Let  $\mathcal{H} = \{H_n\}_n$  be an infinite family of Tanner graphs and  $b : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ . Consider transmitting  $x \in \mathbb{F}_2^n$  and receiving the corrupted version  $y \in \mathbb{F}_2^n$  of  $x$ . We say that the LP decoder of  $H_n$  corrects  $y$  with  $b(n)$  help bits if there exists  $z \in \mathbb{F}_2^n$  of weight at most  $b(n)$  such that the LP decoder of  $H_n$  succeeds in recovering  $x$  from  $y + z$ . That is, we are allowed to flip at most  $b(n)$  bits of  $y$  to help the LP decoder. Define the LP-threshold  $\xi_{LP}(\mathcal{H}, b)$  to be the supremum of  $\epsilon \geq 0$  such that the probability that the LP decoder of  $H_n$  fails with  $b(n)$  help bits over the  $\epsilon$ -BSC tends to zero as  $n$  tends to infinity, i.e.,*

$$\xi_{LP}(\mathcal{H}, b) = \sup\{\epsilon \geq 0 : \Pr_{\epsilon\text{-BSC}}[\text{LP decoder of } H_n \text{ fails with } b(n) \text{ help bits}] = o(1)\}.$$

**Theorem 9.2 (Sublinear help does not improve LP threshold)** *Let  $\mathcal{H} = \{H_n\}_n$  be an infinite family of Tanner graphs. If  $\mathcal{H}$  is asymptotically strong and  $b(n) = o(n)$ , then  $\xi_{LP}(\mathcal{H}, b) = \xi_{LP}(\mathcal{H})$ .*

A potential constructive application of Theorem 9.2 is the following. In dual terms (by Theorem 1.2), the LP decoder of  $H_n = (V_n, C_n, E_n)$  corrects  $y$  with  $b(n)$  help bits iff there is a  $b(n)$ -weak dual witness for  $(-1)^y$ , where  $w : V \rightarrow \mathbb{R}$  is called a  $b(n)$ -weak dual witness if instead of the variable nodes inequalities  $F_i(w) < (-1)^{y_i}$ , for  $i \in V$ , it satisfies the following weaker version:

$$\begin{cases} F_i(w) < 1 & \text{for all } i \in V_n \\ F_i(w) < -1 & \text{for all but at most } b(n) \text{ variable } i \in V_n \text{ such that } y_i = 1. \end{cases}$$

Thus Theorem 9.2 implies that to estimate the LP threshold of an asymptotically strong Tanner graph, it is enough to find a weak dual witness instead of a dual witness, which is in principle an easier task.

The proof of Theorem 9.2 is below and it uses the following converse of the LP excess lemma (Lemma 5.1).

**Lemma 9.3 (LP deficiency lemma: trading LP deficiency with crossover probability)** *Let  $H = (V, C, E)$  be a Tanner graph. Let  $0 < \epsilon < \epsilon' < 1$  and  $0 < \delta < 1$  such that  $\epsilon' = \epsilon + (1 - \epsilon)\delta$ . Let  $q_{\epsilon', \delta}$  be the probability that there is no dual witness in  $H$  for  $(-1)^y + \frac{\delta}{2}$ , where  $y \sim \text{Ber}(\epsilon', n)$  is an error pattern generated by the  $\epsilon'$ -BSC. Then the probability that the LP decoder of  $H$  fails on the  $\epsilon$ -BSC is at most  $\frac{2q_{\epsilon', \delta}}{\delta}$ .*

The proof of the LP deficiency lemma is in Section 9.1. Note that the  $\frac{\delta}{2}$  term in  $(-1)^y + \frac{\delta}{2}$  represents the “LP deficiency” of the dual witness with respect to  $(-1)^y$ , i.e., how far it is from being a dual witness for  $(-1)^y$ .

**Proof of Theorem 9.2:** The proof uses a part of the argument in Theorem 1.2 and applies the LP deficiency lemma instead of the LP excess lemma. Using the asymptotic strength of  $\mathcal{H}$ , we

will trade the help bits with LP deficiency, which in turns we will trade with crossover probability using the LP deficiency lemma. At a high level, the argument is as follows. For any  $\delta > 0$ , we will operate the LP decoder of  $\mathcal{H}$  with  $b(n)$  help bits on the BSC below its threshold  $\xi_{LP}(\mathcal{H}, b)$  by around  $\frac{\delta}{2}$ . With high probability, we have a dual witness  $w$  for  $(-1)^{y+z}$  for some help vector  $z \in \{0, 1\}^n$  of sublinear weight. We will turn  $w$  into a dual witness for  $(-1)^y + \frac{\delta}{4}$  by patching to  $w$  a dual witness  $v$  for the asymmetric LLR vector  $\mu(z, \delta)$  given by

$$\mu_i(z, \delta) = \begin{cases} -2 & \text{if } z_i = 1 \\ \frac{\delta}{4} & \text{if } z_i = 0, \end{cases} \quad (15)$$

for all  $i \in V$ . The existence of  $v$  follows from the asymptotic strength of  $\mathcal{H}$ . Using the LP deficiency lemma, we get rid of the deficiency  $\frac{\delta}{4}$  by decreasing the crossover probability to  $\xi_{LP}(\mathcal{H}, b) - \delta$ .

More precisely, assume without loss of generality that  $\xi_{LP}(\mathcal{H}, b) > 0$  and consider any (small) constant  $0 < \delta < \xi_{LP}(\mathcal{H}, b)$ . We will show that  $\xi_{LP}(\mathcal{H}) \geq \xi_{LP}(\mathcal{H}, b) - \delta$ . Let  $\epsilon = \xi_{LP}(\mathcal{H}, b) - \delta$  and  $\epsilon' = \epsilon + (1 - \epsilon)\frac{\delta}{2}$ , thus  $0 < \epsilon < \epsilon' < \xi_{LP}(\mathcal{H})$ . Let  $q_{\epsilon'}(n)$  be the probability that the LP decoder of  $H_n$  with  $b(n)$  help bits fails on  $(-1)^y$ , where  $y \sim \text{Ber}(n, \epsilon')$ . Since  $\epsilon' < \xi_{LP}(\mathcal{H}, b)$ , we have  $q_{\epsilon'}(n) = o(n)$ . By Theorem 2.2, with probability  $1 - q_{\epsilon'}(n)$  over the choice of  $y \sim \text{Ber}(n, \epsilon')$ , there is a dual witness  $w$  in  $H_n$  for  $(-1)^{y+z}$  for some  $z \in \{0, 1\}^n$  of weight at most  $b(n)$ . Consider any  $n$  and any  $y \in \{0, 1\}^n$  such that  $w$  and  $z$  exist. Since  $\mathcal{H}$  is asymptotically strong, there exists a constant  $\alpha_\delta > 0$  (independent of  $n$ ) such that if  $\text{weight}(z) \leq \alpha_\delta n$ , the LP decoder of  $H_n = (V, C, E)$  succeeds on the asymmetric LLR vector  $\mu(z, \delta)$  defined in (15). Accordingly, by Theorem 2.2, let  $v : E \rightarrow \mathbb{R}$  be a dual witness for  $\mu(z, \delta)$ . Since  $b(n) = o(n)$ , assume that  $n$  is large enough so that  $b(n) \leq \alpha_\delta n$ . It follows that  $w + v$  is a dual witness for  $(-1)^{y+z} + \mu(z, \delta)$ . Since  $(-1)^{y+z} + \mu(z, \delta) \leq (-1)^y + \frac{\delta}{4}$ , we get that  $w + v$  is a dual witness for  $(-1)^y + \frac{\delta}{4}$ . Therefore, the probability that there is no dual witness in  $H_n$  for  $(-1)^y + \frac{\delta}{4}$  over the choice of  $y \sim \text{Ber}(n, \epsilon')$  is at most  $q_{\epsilon'}(n)$ . It follows from the LP deficiency lemma that the probability that the LP decoder of  $H_n$  fails on the  $\epsilon$ -BSC is at most  $\frac{4q_{\epsilon'}(n)}{\delta}$ . Since  $q_{\epsilon'}(n) = o(n)$ , we get that  $\xi_{LP}(\mathcal{H}) > \epsilon = \xi_{LP}(\mathcal{H}, b) - \delta$ .  $\blacksquare$

### 9.1 Proof of Lemma 9.3

The proof is a variation of the argument in the proof of Theorem 8.1 in [BGU14]. Decompose the  $\epsilon'$ -BSC into the bitwise OR of the  $\epsilon$ -BSC and the  $\delta$ -BSC. Choose  $x \sim \text{Ber}(\epsilon, n)$  and  $e'' \sim \text{Ber}(\delta, n)$  and consider  $e = x \vee e''$ , thus  $e \sim \text{Ber}(\epsilon', n)$ . At a high level, we will construct a dual witness on the  $\epsilon$ -BSC by appropriately averaging dual witnesses on the  $\epsilon'$ -BSC over the choice of  $e'' \sim \text{Ber}(\delta, n)$ .

For every  $y \in \{0, 1\}^n$ , let

$$L(y) = \begin{cases} 1 & \text{if } (-1)^y + \frac{\delta}{2} \text{ has a dual witness} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, in terms of  $L$ ,

$$q_{\epsilon', \delta} = \Pr_{y \sim \text{Ber}(\epsilon', n)} [L(y) = 0]. \quad (16)$$

If  $y \in \{0, 1\}^n$ , let  $v^y : E \rightarrow \mathbb{R}$  be an arbitrary dual witness for  $(-1)^y + \frac{\delta}{2}$  if  $L(y) = 1$ . Otherwise, let  $v^y : E \rightarrow \mathbb{R}$  be the identically zero function. If  $x \in \{0, 1\}^n$ , define  $w^x : E \rightarrow \mathbb{R}$  by averaging  $v^{x \vee e''}$  over the choice of  $e'' \sim \text{Ber}(\delta, n)$  and scaling:

$$w^x = \alpha \mathbb{E}_{e'' \sim \text{Ber}(\delta, n)} v^{x \vee e''},$$

where  $\alpha = \frac{1}{1-\delta} > 0$ . We will show that  $w^x$  is a dual witness for  $(-1)^x$  with probability at least  $1 - \frac{2q_{\epsilon',\delta}}{\delta}$  over the choice of  $x \sim \text{Ber}(\epsilon, n)$ .

If  $L(y) = 1$ , then by definition,  $v^y$  satisfies the dual witness check nodes inequalities:  $v^y(i, j) + v^y(i', j) \geq 0$ , for each check  $j \in C$  and all distinct variables  $i \neq i' \in N(j)$ . The identically zero function  $E \rightarrow \mathbb{R}$  also satisfies those inequalities, hence they are satisfied by  $v^y$  for all  $y \in \{0, 1\}^n$ . Since  $w^x$  is an average over  $v^{x \vee e''}$  scaled by a positive constant, we get that the dual witness check nodes inequalities are satisfied by  $w^x$  for all  $x \in \{0, 1\}^n$ .

In what follows, we take care of the variable nodes inequalities ((a) in Definition 2.1). If  $w : E \rightarrow \mathbb{R}$ , consider the flow vector  $F(w) \in \mathbb{R}^V$  associated with  $w$ :  $F_i(w) = \sum_{j \in N(i)} w(i, j)$ , for all  $i \in V$ . In terms of  $F$ , we have

$$F(v^y) < (-1)^y + \frac{\delta}{2}, \text{ for each } y \in \{0, 1\}^n \text{ such that } L(y) = 1. \quad (17)$$

We have to show that  $F(w^x) < (-1)^x$  with probability at least  $1 - \frac{2q_{\epsilon',\delta}}{\delta}$  over the choice of  $x \sim \text{Ber}(\epsilon, n)$ . For any  $x \in \{0, 1\}^n$ ,

$$\begin{aligned} F(w^x) &= \alpha \mathbb{E}_{e'' \sim \text{Ber}(\delta, n)} F(v^{x \vee e''}) \\ &= \alpha \mathbb{E}_{e''} [ F(v^{x \vee e''}) \mid L(x \vee e'') = 1 ] \times \Pr_{e''} [L(x \vee e'') = 1] \\ &< \alpha \mathbb{E}_{e''} [ (-1)^{x \vee e''} + \frac{\delta}{2} \mid L(x \vee e'') = 1 ] \times \Pr_{e''} [L(x \vee e'') = 1] \quad (\text{using (17)}) \\ &= \alpha \left( \mathbb{E}_{e''} (-1)^{x \vee e''} + \frac{\delta}{2} - \mathbb{E}_{e''} [ (-1)^{x \vee e''} \mid L(x \vee e'') = 0 ] \times \phi_x \right) \\ &\leq \alpha \left( \mathbb{E}_{e''} (-1)^{x \vee e''} + \frac{\delta}{2} + \phi_x \right) \end{aligned} \quad (18)$$

where  $\phi_x := \Pr_{e'' \sim \text{Ber}(\delta, n)} [L(x \vee e'') = 0]$ . Note that (18) follows from the fact that  $L(y) = 0$  implies  $v^y = 0$  and hence  $F(v^y) = 0$ . Fix any  $i \in V$ . If  $x_i = 1$ , then  $\mathbb{E}_{e''} (-1)^{x_i \vee e''_i} = -1$ . If  $x_i = 0$ , then  $\mathbb{E}_{e''} (-1)^{x_i \vee e''_i} = \delta(-1) + (1 - \delta)(1) = 1 - 2\delta$ . Hence

$$F_i(w^x) < \begin{cases} \alpha(-1 + \frac{\delta}{2} + \phi_x) & \text{if } x_i = 1 \\ \alpha(1 - \frac{3\delta}{2} + \phi_x) & \text{if } x_i = 0. \end{cases}$$

By (16),

$$\mathbb{E}_{x \sim \text{Ber}(\epsilon, n)} \phi_x = \Pr_{e'' \sim \text{Ber}(\delta, n), x \sim \text{Ber}(\epsilon, n)} [L(x \vee e'') = 0] = \Pr_{y \sim \text{Ber}(\epsilon', n)} [L(y) = 0] = q_{\epsilon', \delta}.$$

Thus, by Markov's inequality,  $\phi_x \geq \frac{\delta}{2}$  with probability at most  $\frac{2q_{\epsilon',\delta}}{\delta}$  over the choice  $x \sim \text{Ber}(\epsilon, n)$ . Hence, with probability at least  $1 - \frac{2q_{\epsilon',\delta}}{\delta}$  over  $x \sim \text{Ber}(\epsilon, n)$ , we have for all  $i \in V$ ,

$$\begin{aligned} F_i(w^x) &< \begin{cases} \alpha(-1 + \frac{\delta}{2} + \frac{\delta}{2}) & \text{if } x_i = 1 \\ \alpha(1 - \frac{3\delta}{2} + \frac{\delta}{2}) & \text{if } x_i = 0. \end{cases} \\ &= (-1)^{x_i}, \end{aligned}$$

since  $\alpha = \frac{1}{1-\delta}$ .

## 10 Discussion and open problems

We conclude with some remarks and open questions mainly related to the asymptotic strength condition, the rigidity condition and the LP decoding threshold on the BSC.

**Asymptotic strength condition.** Theorem 1.3 shows that expansion implies asymptotic strength. We know that random low density Tanner graphs are good expanders with high probability [SS96, FMS+07]. Combining Theorem 1.3 and the probabilistic analysis in Appendix B of [FMS+07] implies the following.

**Theorem 10.1** *Let  $0 < r < 1$  be a constant. Let  $d_v$  be a positive integer constant such that there exists a constant  $\frac{2}{3} < \delta < 1$  for which  $\delta d_v$  and  $(1 - \delta)d_v$  are integers and  $(1 - \delta)d_v \geq 2$ . Then, for any positive integers  $n$  and  $m$  such that  $r = 1 - \frac{m}{n}$ , a random variable-regular Tanner graph  $G$  with variable degree  $d_v$ ,  $n$  variable nodes and  $m$  check nodes is asymptotically strong with high probability<sup>1</sup>.*

The integrality constraint on  $\delta d_v$  and  $(1 - \delta)d_v$  can require large values of  $d_v$  (see [FMS+07]). We conjecture that the following holds.

**Conjecture 10.2** *For all  $d_c > d_v \geq 3$ , a random  $(d_v, d_c)$ -regular Tanner graph is asymptotically strong with high probability.*

**Rigidity condition.** If the graph has  $\Theta(\log n)$  girth and minimum check degree at least 3, the rigidity condition is equivalent to the simpler  $(c \log n, \omega(1))$ -nondegeneracy condition. We argued in Lemma 1.10 that the latter condition holds with high probability for random check-regular graphs assuming that  $m < \beta_d n$ , where  $m$  is the number of checks nodes,  $d$  is the check degree and  $\beta_d$  is Calkin's threshold. The statistical independence of the check nodes in the ensemble of random check-regular graphs makes the ensemble attractive from a probabilistic analysis perspective, but it typically gives irregular graphs with constant girth. We believe that good girth and variable-regularity do not increase the odds of degeneracy; we conjecture that nondegeneracy is also a typical property of the ensemble of regular  $\Theta(\log n)$ -girth Tanner graphs.

**Conjecture 10.3** *Let  $d_c > d_v \geq 3$  be integers such that  $d_v < \beta_{d_c} d_c$ . If  $\lambda > 0$  be a constant, let  $\Gamma_\lambda$  be ensemble of  $(d_v, d_c)$ -regular Tanner graphs on  $n$  variable nodes of girth at least  $\lambda \log n$ . Then there is a constant  $\lambda > 0$  small enough such that for each constant  $c > 0$ , a random graph  $G$  from the ensemble  $\Gamma_\lambda$  is  $(c \log n, \omega(1))$ -nondegenerate with high probability.*

Establishing this conjecture requires working in a more complex probabilistic framework. We leave the question open for further investigation. Note that since  $d_c m = d_v n$ , the condition  $d_v < \beta_{d_c} d_c$  is equivalent to  $m < \beta_{d_c} n$ . A natural but probably more difficult problem is to study also the asymptotic strength of the ensemble  $\Gamma_\lambda$ .

**Limits of LP decoding on the BSC.** On the positive side, our negative results suggest studying

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<sup>1</sup> By a random family  $\mathcal{G} = \{G_n\}_n$  of Tanner graphs being asymptotically strong with high probability, we mean the following. For each constant  $\beta > 0$ , there exists a constant  $\alpha > 0$  such that for each  $n$ , with probability at least  $1 - o(1)$  over the random choice of  $G_n$ , the LP decoder of  $G_n$  succeeds on the asymmetric LLR vector  $\gamma(y, \beta)$  for all  $y \in \{0, 1\}^n$  of weight at most  $\alpha n$ .

the LP decoding limits in the framework of the dual code containing all redundant checks. This framework is appealing since it is independent of a particular Tanner graph representation of the code. If  $r'$  is the *rate of the dual code*, Shannon's limit says that we can transmit reliably over the  $\epsilon$ -BSC if  $\epsilon < H^{-1}(r')$ , where  $H$  is the binary entropy function. For LP decoding with all redundant checks included, it is natural to study the following LP capacity function.

**Definition 10.4 (LP capacity over the BSC)** *Given a dual rate  $0 < r' < 1$ , define the LP capacity function*

$$\xi_{LP}(r') := \sup_{\{D_n\}_n} \xi_{LP}(\{G_{D_n}\}_n),$$

where the supremum is over all  $\mathbb{F}_2$ -linear codes  $D_n \subset \mathbb{F}_2^n$  such that  $\lim_{n \rightarrow \infty} \text{rate}(D_n) = r'$  and  $G_{D_n}$  is the Tanner graph on  $n$  variables whose checks are the nonzero elements of  $D_n$ .

Note that primitive hyperflows (Theorem 4.2) maybe useful in studying the LP capacity function.

**Question 10.5** *i) (Relation to Shannon's capacity) How far is  $\xi_{LP}(r')$  from the Shannon's capacity  $H^{-1}(r')$ ? Is  $\xi_{LP}(r') = H^{-1}(r')$ ?*

*ii) (Achievability with bounded check-degree) Is any  $\epsilon < \xi_{LP}(r')$  achievable by a family of codes  $\{D_n\}_n$  with a bounded-weight basis? That is, is it true that for each  $\epsilon < \xi_{LP}(r')$ , there exist a constant  $d$  and a family of Tanner graphs  $\mathcal{G} = \{G_n\}_n$  such that  $\xi_{LP}(\overline{\mathcal{G}}) \geq \epsilon$  and  $G_n$  has at most  $r'n$  check nodes each of degree most  $d$ ?*

*iii) (Achievability with asymptotic strength) If the answer of (ii) is affirmative, is  $\mathcal{G}$  asymptotically strong?*

*iv) (Achievability with rigidity) If the answer of (iii) is affirmative, is  $\mathcal{G}$  rigid?*

The answer to first question is not clear.

The answer to (ii) is probably affirmative since we already know from [FMS+07] that a positive value of  $\epsilon$  is achievable with bounded check degree. The answer to (iii) seems also affirmative. In general, asymptotic strength makes the LP stronger as it guarantees that the fractional weight spectrum of the pseudocodewords is not “too unbalanced” (Theorem 8.1). Inspired by [LMSS01], if  $\mathcal{G}$  is not asymptotically strong, we can actually make it asymptotically strong without noticeably affecting its rate by adding to the code a small number of parity checks which form a sufficiently good expander<sup>2</sup>. The added checks do not decrease the LP threshold of the code.

If both (ii) and (iii) have affirmative answers, we obtain from Theorem 1.2 that for any  $\epsilon < \xi_{LP}(r')$ , there exists a sufficiently large constant  $k \geq d$  such that  $\xi_{LP}(\overline{\mathcal{G}}^k) \geq \epsilon$ . Thus, by running the LP decoder of  $\overline{\mathcal{G}}^k$ , dual rate  $r'$  is achievable on the  $\epsilon$ -BSC in time polynomial in the block length  $n$ . More specifically, the time is polynomial in  $n^k$ , where the constant  $k$  increases as the gap  $\delta = \xi_{LP}(r') - \epsilon$  gets small.

The last question is more intriguing. If the answer to (iv) is also affirmative, then  $\xi_{LP}(\overline{\mathcal{G}}) = \xi_{LP}(\mathcal{G})$  by Corollary 1.7. Thus, by running the LP decoder of  $\mathcal{G}$ , we conclude that for any  $\epsilon < \xi_{LP}(r')$ , dual rate  $r'$  is achievable on the  $\epsilon$ -BSC in time polynomial in the block length  $n$  and independent of the gap  $\delta$ , which is counter intuitive if  $\xi_{LP}(r') = H^{-1}(r')$ .

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<sup>2</sup>We need a  $(\epsilon n, \delta d')$ -expander between  $n$  variable nodes and  $\alpha n$  check nodes of regular variable degree  $d'$  and bounded check degree, where  $\alpha > 0$  is a small constant and  $\epsilon, \delta > 0$  are constants such that  $\frac{2}{3} < \delta < 1$  and  $\delta d'$  is an integer.

**AWGN.** On a final note, a natural question is to explore the potential extendability of the results in this paper to other channels such as the AWGN.

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